Advanced Higher Maths with Miss Deely

Monday Period 5 and Wednesday Period 7

Topics:
1. Binomial Theorem.
2. Partial Fractions.
3. Complex Numbers
4. Systems of Equations: Gaussian Elimination & Matrix Operations.
5. Sequences and Series.
6. Summation Formulae and Proof by Induction.
7. Vectors.
8. Disproving conjectures by counter-examples.
9. Indirect or direct proofs.

Reminder of Sets and Notation

N - Natural numbers

W - Whole Numbers

Z - Integers

Q - Rational Numbers

R - Real Numbers

'∈' means an element of

' ' means not an element of

means 'therefore'

means 'this implies'

≈ means 'approximately equal to'

'.' can be used instead of x for multiplication.

Σ means the sum of

Factorial Notation

A product of all natural numbers less than or equal to n can be written as n! where '!' means factorial.

For example:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

In general, $n! = n \times (n-1) \times (n-2) \ldots (n-(n-1))$ for n N

We can also say that $n \times (n-1)! = n!$ because $7 \times 6! = 7!$

Note that 1! = 1 and that 0! = 1.

This is beacuse 1 x 0! = 1!

Factorial Notation

Work out the value of the following:

(i) 5! = 120

(ii) 8! = 40320

(iii) 20 x 19! = 20! 2.43.. x 10^{18}

(iv) $\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 20$

(v) $\frac{8!}{6!} = 8 \times 7 = 56$

Permutations

Factorial Notation is used in probability to calculate the number of ways a set of objects/numbers can be arranged/ordered. These are known as permutations.

Q. Think about the letters A and B. How many ways can you order them?

AB

BA

Q. Now, think about the letters A, B and C. How many ways can you order them?

ABC	BCA
ACB	CAB
BAC	CBA

There are 6 ways which is 3 x (number ways you can order 2 letters).

Q. Now, think about the letters A, B, C and D. How many ways can you order them?

ABC D	ACB D	ADB C
BACD	CABD	DABC
BCAD	CBAD	DBAC
BCDA	CBDA	DBCA

etc.

There are 24 ways which is 4 x (number ways you can order 3 letters).

Permutations

So there are n! ways of ordering n objects.

We can then look at 0! again. How many ways are there of ordering an empty set?

{} = 1 way, therefore 0! = 1

Permutations

If there are n objects and we want to choose r objects to order at a time, we can divide factorials.

For example: If we have A,B,C,D, and we want to order 2 letters

AB AC AD BC BD CD

BA CA DA CB DB DC

There are 4 choices for the first space and 3 choices for the second.

So there are 4 x 3 = 12 choices

Q. How can you write this using factorial notation?

Q. What if you have 5 letters and want to order 3?

In this case, anything after 3! is not needed so to cancel it out so we can say 4! ÷ 2! = 12

$$\frac{4!}{2!} = \frac{4\times3\times2\times1}{2\times1} = 12$$

Permutations

Algebraically, this can be written as $\frac{n!}{(n-r)!}$

where n is the number of objects. 'r' is how many you are choosing.

We write it in this way so we can solve for unknowns and think in general terms.

On your calculator, you can use the ^nP_r button.

$$\frac{6!}{(6-3)!} = \frac{6!}{3!} \qquad \frac{n!}{(n-r)!}$$

Permutations Starter

$$\frac{n!}{(n-r)!}$$

Work out the following:

1. How many ways can you order 3 books on a shelf from a choice of 7 books?

$$\frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$$

2. In how many ways can the letters of the word BRIDGE be arranged in a row taking 4 at a time?

$$\frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$$

Today we are going to learn about Combinations.

Combinations

If you are given n objects, you have to place r of them and the order doesn't matter, there will be less possibilities. These are known as combinations.

To work out combinations, we first look at all the permutations and then get rid of the unnecessary variations.

Combinations

$\frac{n!}{(n-r)!}$

For example: We have to choose 3 letters from A, B, C, D, E and the order doesn't matter.

How can you write this using factorial notation? $\frac{5!}{3!(5-3)!}$

Can we write a general formula to show it algebraically?

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Combinations

In general, we write it algebraically as $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Examples:

1. How many ways can you choose 3 friends from 10 friends to go to a concert?

$$\frac{10!}{3!(10-3)!} = \frac{10!}{3!\,7!} = \frac{10\times9\times8\times7!}{3!\,7!} = 120$$

2. The lotto! How many ways is there of choosing 6 numbers from 59?

$${}^nC_R = {}^{59}C_6 = 45\ 057\ 474$$

Combinations

There are some statements we can now consider –

$\binom{n}{n}=1$ $\quad \binom{n}{1}=n$ $\quad \binom{n}{0}=1$ $\quad \binom{n}{r}=\binom{n}{n-r}$

Try to think about why these are true

$\binom{6}{2}$ $\quad \binom{6}{4}$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!\,r!}$$

Combinations

Can you prove the following statement? $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

L.H.S

$$\frac{(n+1)!}{r!(n+1-r)!}$$

R.H.S

$$\frac{n!}{(r-1)!(n-(r-1))!} + \frac{n!}{r!(n-r)!}$$

$$\frac{n!\times r}{(r-1)!(n-r+1)!\times r} + \frac{n!\times(n-r+1)}{r!(n-r)!\times(n-r+1)}$$

$$\frac{n!\,r}{r!(n-r+1)!} + \frac{n!(n-r+1)}{r!(n-r+1)!}$$

$$\frac{n!\,r + n!(n-r+1)}{r!(n-r+1)!}$$

$$\frac{n!(r + n - r + 1)}{r!(n-r+1)!}$$

$$\frac{n!(n+1)}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = \text{L.H.S}$$

Using our notation to solve for 'n' and/or 'r'

Examples: Solve for n, where n N $\qquad \binom{n}{r} = \frac{n!}{r!(n-r)!}$

1. $\binom{n+1}{2} = 45$

$$\Rightarrow \frac{(n+1)!}{2!(n+1-2)!} = \frac{(n+1)!}{2(n-1)!} = \frac{(n+1)(n)(n-1)!}{2(n-1)!} = 45$$

$\times 2 \quad \times 2$

$$n(n+1) = 90$$

2. $\binom{n+1}{2} + \binom{n}{1} = 35$

$$\frac{(n+1)!}{2!((n+1)-2)!} + n = 35$$

$$\frac{(n+1)!}{2(n-1)!} + n = 35$$

$$\frac{(n+1)(n)(n-1)!}{2(n-1)!} + n = 35$$

$$\frac{(n+1)(n)}{2} + n = 35$$

$$\frac{n^2+n}{2} + n = 35$$

$$\frac{n^2+n}{2} + \frac{2n}{2} = 35$$

$$\frac{n^2+3n}{2} = 35$$

$$n^2+3n = 70$$

$$n^2+3n-70 = 0$$

$$(n+10)(n-7) = 0$$

$$n=7$$

Q1. Prove that $\frac{(n+5)!}{(n+3)!} = n^2 + 9n + 20$

$$\frac{(n+5)(n+4)(n+3)!}{(n+3)!}$$

$$(n+5)(n+4)$$

$$n^2+9n+20$$

Q2. Solve the following equations where $n \in N$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

(i) $\binom{n-1}{2} = 28$ (ii) $2!\,(n-2)! = 3!\,(n-3)!$

$\frac{(n-1)!}{2!(n-1-2)!} = 28$

$\frac{(n-1)!}{(n-3)!} = 56$

$\frac{(n-1)(n-2)(n-3)!}{(n-3)!} = 56$

$(n-1)(n-2) = 56$

$n^2 - 3n - 54 = 0$

$(n+6)(n-9) = 0$

$n = 9$

(ii) $2!\,(n-2)! = 3!\,(n-3)!$

$2(n-3)!(n-2) = 6(n-3)!$

$2n - 4 = 6$

$2n = 10$

$n = 5$

$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

(ii) $\binom{4}{n-1} + \binom{4}{n} = 5$

$\binom{4+1}{n} = 5$

$\binom{5}{n} = 5$

$n = 4$ or $n = 1$

Now we are moving on to the other aspect of Binomial Expansion.

Expanding Brackets

Multiply out the following and simplify:

(i) $(x + y)^2$

(ii) $(x + y)^3$

(iii) $(x + y)^4$

(iv) $(x + y)^5$

Pascal's Triangle

We notice that a pattern starts to emerge when we multiply out brackets to a power.

$x + y$

$x^2 + 2xy + y^2$

$x^3 + 3x^2y + 3xy^2 + y^3$

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Pascal's Triangle

The numbers in Pascal's triangle represent the coefficients of the terms in the expansions.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Increasing powers of y →

Decreasing powers of x

Pascal's Triangle

The coefficients can be written using $\binom{n}{r}$ notation.

They are the same as working out the combination when n is the power and r is ascending from 0 to n.

$(x+y)^n$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

The Binomial Theorem

Using Pascal's Triangle, we can write that in general

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 \ldots\ldots + \binom{n}{n}a^{n-n}b^n$$

$$(a+b)^n = \sum_{r=0}^{n} a^{n-r}$$

The Binomial Theorem

We can write a simplified version using Sigma Notation

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

This formula is provided in your exam.

The Binomial Theorem

Examples: Expand the following using the Binomial Theorem

$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$

1. $(x + 2y)^4$

$\binom{4}{0}x^4(2y)^0 + \binom{4}{1}x^3(2y)^1 + \binom{4}{2}x^2(2y)^2 + \binom{4}{3}x(2y)^3 + \binom{4}{4}x(2y)^4$

$x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

2. $(1 - 3b)^3$

$\binom{3}{0}1^3(-3b)^0 + \binom{3}{1}1^2(-3b) + \binom{3}{2}1(3b)^2 + \binom{3}{3}1^0(3b)^3$

$1 - 9b + 27b^2 - 27b^3$

③ $(2x+3y)^4$

$\binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(3y) + \binom{4}{2}(2x)^2(3y)^2 + \binom{4}{3}(2x\ldots$
$+ \binom{4}{4}(3y)^4$

The Binomial Theorem Homework

1. Expand:

(a)	$(x+5)^3$	(b)	$(a+2)^4$	(c)	$(m-3)^5$
(d)	$(5-y)^4$	(e)	$(2x+3)^3$	(f)	$(3p+1)^4$
(g)	$(2x+y)^5$	(h)	$(2k-5)^3$	(i)	$(3x-2y)^4$
(j)	$(x^2+2)^3$	(k)	$(y^3-2)^4$	(l)	$\left(x+\frac{1}{x}\right)^3$

pg.9 choose some from q.1

$(x+x^{-1})^3$

(l) $\binom{3}{0}x^3 + \binom{3}{1}x^2(x^{-1}) + \binom{3}{2}x(x^{-1})^2 + \binom{3}{3}(x^{-1})^3$

$x^3 + 3x + 3x^{-1} + x^{-3}$

$x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

The Binomial Theorem

3. Find the 4th term in the expansion of $(x^2 - \frac{2}{x})^7$ $= (x^2 - 2x^{-1})^7$

$\binom{7}{3}(x^2)^4(-2x^{-1})^3$

$35x^8(-8x^{-3})$

$-280x^5$

$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$

Challenge Question: Homework

If the first 3 terms in the expansion of $(1+ax)^n$ are $1-4x+7x^2$, find the values of a and n, for $n \in \mathbb{N}, n>1$

$-4x = \binom{n}{1}ax$

$-4x = nax$

$-4 = na$

$a = \frac{-4}{n}$

$7x^2 = \binom{n}{2}\left(\frac{-4x}{n}\right)^2$

$= \binom{n}{2}\frac{16x^2}{n^2}$

$7n^2 = \binom{n}{2}16$

$\frac{7n^2}{16} = \binom{n}{2}$

$\frac{7n^2}{16} = \frac{n!}{2!(n-2)!}$

$\frac{7n^2}{16} = \frac{n(n-1)(n-2)!}{2(n-2)!}$

$\frac{7n^2}{16} = \frac{n(n-1)}{2}$

$\frac{7n^2}{8} = n^2 - n$

$7n^2 = 8n^2 - 8n$

$0 = n^2 - 8n$

$0 = n(n-8)$

$n = 8$

$\underline{n=8}$

$-4 = na$

$-4 = 8a$

$a = -\frac{1}{2}$

The Binomial Theorem

Finding the term independent of a or b

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

We can substitute in the value of r

The Binomial Theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

Examples:

1. Find the coefficient of the term involving x^5 in the expansion $(4+2x)^8$.

$(4+2x)^8 = \sum_{r=0}^{8}\binom{8}{r}(4)^{8-r}(2x)^r$

Let $r=5$

$\binom{8}{5}(4)^{8-5}(2x)^5$

$(56)(64)(32x^5)$

$= 114688x^5$

Coefficient $= \underline{114688}$

The Binomial Theorem

2. Find the term independent of x in the binomial expansion of $(3x - \frac{2}{x^2})^{18}$.

$(3x-2x^{-2})^{18}$

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

$\sum_{r=0}^{18}\binom{18}{r}(3x)^{18-r}(2x^{-2})^r$

$(3x)^{18-r}(2x^{-2})^r$

$(x^{18-r})(x^{-2r}) = x^0$

$18-r+(-2r)=0$

$18-3r=0$

$\underline{r=6}$

$\binom{18}{6}(3x)^{18-6}(2x^{-2})^6$

$(18564)(531441x^{12})(64x^{-12})$

$\underline{6.31 \times 10^{11}\ (2\text{d.p.})}$

The Binomial Theorem — PP Questions

Q. State and simplify the general term in the binomial expansion of $\left(2x-\frac{5}{x^2}\right)^6$.
Hence, or otherwise, find the term independent of x.

Q. 1. Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3}-\frac{2}{x}\right)^5.$$

Q. Write down and simplify the general term in the expansion of $\left(2x-\frac{1}{x^2}\right)^9$.
Hence, or otherwise, obtain the term independent of x.

Q. Write down and simplify the general term in the expression $\left(\frac{2}{x}+\frac{1}{4x^2}\right)^{10}$.

Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$.

Q. Write down the binomial expansion of $\left(3x-\frac{2}{x^2}\right)^4$ and simplify your answer. 4

Q. Use the binomial theorem to expand $\left(\frac{1}{2}x-3\right)^4$ and simplify your answer. 3

Q. Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3}-\frac{2}{x}\right)^5.$$

The Binomial Theorem

Expanding multiple brackets

Use the Binomial Theorem on each separately

Example: Expand out $(1+x-x^2)(1-x)^8$ up to and including the term x^5.

The Binomial Theorem

2. What is the coefficient of x^5 in the expansion of $(1 + x)^4(1 - 2x)^5$?