The Euclidean Algorithim

An algorithim is a step-by-step procedure for performing a calculation according to well-defined rules.

The Euclidean Algorithim is an efficient method to find the greatest common divisor (aka the highest common factor) of a pair of numbers.

Two integers are said to be coprime or relatively prime when the g.c.d or h.c.f is 1.

The Euclidean Algorithim

When you are finding the gcd of a pair of numbers a and b, use the notation (a, b).

For example:

(6, 9) = 3

The Euclidean Algorithim

First express one number in terms of the other plus its remainder.

Examples: Use the Euclidean Algorithim for the following:

(a) Find the gcd of 140 and 252

$252 = 1 \times 140 + 112$

$140 = 1 \times 112 + 28$

$112 = 4 \times 28 + 0$

The last non-zero remainder is the g.c.d

$(140, 252) = 28$

The Euclidean Algorithim

The remainder is zero and the greatest common divisor is th alast non-zero remainder.

The Euclidean Algorithim

Examples: Use the Euclidean Algorithim for the following:

(b) Find the gcd of 132 and 424

$424 = 3 \times 132 + 28$

$132 = 4 \times 28 + 20$

$28 = 1 \times 20 + 8$

$20 = 2 \times 8 + 4$

$8 = 2 \times 4 + 0$

$(132, 424) = 4$

The Euclidean Algorithim

Examples: Use the Euclidean Algorithim for the following:

(c) Find the gcd of 280 and 117

$280 = 2 \times 117 + 46$

$117 = 2 \times 46 + 25$

$46 = 1 \times 25 + 21$

$25 = 1 \times 21 + 4$

$21 = 5 \times 4 + 1$

$4 = 4 \times 1 + 0$

$(280, 117) = 1$

The Euclidean Algorithm

Examples:

(d) Simplify $\dfrac{684}{1692}$

$1692 = 2 \times 684 + 324$

$684 = 2 \times 324 + 36$

$324 = 9 \times 36 + \underline{\underline{0}}$

gcd = 36

$\dfrac{684 \div 36}{1692 \div 36} = \dfrac{19}{\underline{\underline{47}}}$

Writing the gcd as a linear combination

This is the process of finding integers x and y such that (a, b) = xa + yb

Examples:

1. (a) Use the Euclidean algorithim to show that (231, 17) = 1

(b) Hence find integers such that 231x + 17y = 1

(a)

$231 = 13 \times 17 + 10$

$17 = 1 \times 10 + 7$

$10 = 1 \times 7 + 3$

$* \; 7 = 2 \times 3 + \underline{\underline{1}}$

$3 = 3 \times 1 + 0$ gcd 231 and 17 = 1

(b)

$* \; 1 = 7 - (2 \times 3)$

$1 = 7 - (2 \times (10 - 1 \times 7))$

$1 = 7 - (2 \times 10 - 2 \times 7)$

$1 = 7 - 2 \times 10 + 2 \times 7$

$1 = 3 \times 7 - 2 \times 10$

$1 = 3 \times (17 - 1 \times 10) - 2 \times 10$

$1 = 3 \times 17 - 3 \times 10 - 2 \times 10$

$1 = 3 \times 17 - 5 \times 10$

$1 = 3 \times 17 - (5 \times (231 - 13 \times 17))$

$1 = 3 \times 17 - 5 \times 231 + 65 \times 17$

$1 = 68 \times 17 - 5 \times 231$

$\underline{\underline{x = -5}} \quad \underline{\underline{y = 68}}$

Writing the gcd as a linear combination

2. Find integers x and y such that 1078x + 420y = 14

(1078, 420)

$1078 = 2 \times 420 + 238$

$420 = 1 \times 238 + 182$

$238 = 1 \times 182 + 56$

$* \; 182 = 3 \times 56 + \underline{14}$

$56 = 4 \times 14 + 0$

Working backwards

$14 = 182 - 3 \times 56$

$14 = 182 - 3 \times (\cancel{238 - 1 \times 182})$

$14 = 182 - 3 \times 238 + 3 \times 182$

$14 = 4 \times 182 - 3 \times 238$

$14 = 4 \times (\cancel{420 - 1 \times 238}) - 3 \times 238$

$14 = 4 \times 420 - 4 \times 238 - 3 \times 238$

$14 = 4 \times 420 - 7 \times 238$

$14 = 4 \times 420 - 7 \times (\cancel{1078 - 2 \times 420})$

$14 = 4 \times 420 - 7 \times 1078 + 14 \times 420$

$14 = 18 \times 420 - 7 \times 1078$

$\underline{\underline{x = -7}} \quad \underline{\underline{y = 18}}$

Use the Euclidean algorithm to find integers p and q such that

$$3066p + 713q = 1. \qquad 4$$

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The Division Algorithm

Any division can be written in the form of an equation

a ÷ b or a/b can be written as a = qb + r

where q is the quotient, b is the divisor and r is the remainder.

Using the Division Algorithm for number bases

We can use the division algorithm to convert integers into other number bases such as binary, octal and hexadecimal.

We currently use the base 10 system

25367

10^5	10^4	10^3	10^2	10^1	10^0
	2	5	3	6	7

$2 \times 10^4 + 5 \times 10^3 + 3 \times 10^2 + 6 \times 10^1 + 7 \times 10^0 = 25367$

We can use any integer as a number base but the most useful are 2, 8, and 16.

The divisor is always the number base you are converting to.

Using the Division Algorithm for number bases

Base 2 – Binary

This is a system which involves only two digits; 0 and 1. The place values are all powers of 2.

$$\begin{array}{ccccc} 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 1 & 1 & 0 & 1 & 0 \end{array}$$

$$0\times2^0 + 1\times2^1 + 0\times2^2 + 1\times2^3 + 1\times2^4 = \underline{26}$$

Example: Convert 30 into binary

$30 = 15\times2 + 0$
$15 = 7\times2 + 1$
$7 = 3\times2 + 1$
$3 = 1\times2 + 1$
$1 = 0\times2 + 1$

$30_2 = \underline{11110}$

Using the Division Algorithm for number bases

Base 8 – Octal

This system uses powers of 8 as place values. It only uses the integers 0 to 7.

Example: Convert 53467 into octal form

$53467 = 6683\times8 + 3$
$6683 = 835\times8 + 3$
$835 = 104\times8 + 3$
$104 = 13\times8 + 0$
$13 = 1\times8 + 5$
$1 = 0\times8 + 1$

$\underline{150333}$

Using the Division Algorithm for number bases

Base 16 – Hexadecimal

This system has 16 digits. Since each digit can only occupy one place value, a slight modification is needed. Integers 0-9 are used and then A for 10, B for 11, C for 12, D for 13, E for 14 and F for 15.

Example: Convert 298 047 into hexadecimal

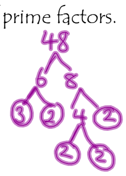
$298\,047 = 18627\times16 + 15$
$18627 = 1164\times16 + 3$
$1164 = 72\times16 + 12$
$72 = 4\times16 + 8$
$4 = 0\times16 + 4$

$\underline{48C3F}$

Fundamental Theorem of Arithmetic

This theorem states that every positive integer > 1 can be written as a product of prime factors.

E.g. 48 =



$48 = 3\times2\times2\times2\times2 = 2^4\times3$

This is known as the canonical form of an integer.

Eg. 1575

$\underline{3^2\times5^2\times7}$

A composite number is one that has factors other than itself and 1.

The Goldbach conjecture states that every even integer greater than 2 is the sum of two primes. (Still has to be proved!)

Specimen Paper

Show that the greatest common divisor of 487 and 729 is 1.

Hence find integers x and y such that $487x + 729y = 1$. 4

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