Starter 25.1.2017

The first and fourth terms of a geometric series are 2048 and 256 respectively. Calculate the value of the common ratio. 2

Given that the sum of the first n terms is 4088, find the value of n. 3

$a = 2048$

$ar^3 = 256$

$r^3 = \frac{256}{a}$

$r = \frac{1}{2}$

$S_n = 4088$

$\frac{a(1-r^n)}{1-r} = 4088$

$2048(1-(\frac{1}{2})^n) = 4088 - 4088(\frac{1}{2})$

$2048 - 2048(2^{-n}) = 2044$

$-2048(2^{-n}) = -4$

$2^{-n} = \frac{1}{512}$

$\frac{1}{2^n} = \frac{1}{2^9}$ $n=9$ or use log!

Power Series

A series in which its terms are increasing powers of x is known as a power series.

For example: $1 + x + x^2 + x^3 + x^4 + \ldots$

What is r in this series? $r = x$

$\frac{a}{1-r}$

If $-1 < x < 1$, this series has a sum to infinity, we can write this as $\frac{1}{1-x}$

Hence S_∞ of $1 + x + x^2 + x^3 + x^4 + \ldots = (1-x)^{-1}$

Power Series

In general, we can express any function ? as a power series in the following way where a_0, a_1 ... are constants

$$\sum_{n=0}^{\infty} a_n x^n = a_0x^0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

A power series may only converge for certain values of x.

Maclaurin Series

Given the series $f(x) = a_0x^0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$ where a_0, a_1 ... are constants

$a_0 + a_1x + a_2x^2 + \ldots$

We can find the coefficients of the above by evaluating the function and its derivatives at x = 0

$f(0) = a_0 + a_1(0) + a_2(0)^2 + \ldots$

$f(0) = a_0$

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots$

$f'(0) = a_1 + 2a_2(0) + 3a_3(0)^2$

$f'(0) = a_1$

$f''(x) = 2a_2 + 6a_3x + \ldots$

$f''(0) = 2a_2 + 6a_3(0) + \ldots$

$f''(0) = 2a_2$

$\frac{f''(0)}{2} = a_2$

$f'''(x) = 6a_3 + \ldots$

$f'''(0) = 6a_3$

$\frac{f'''(0)}{6} = a_3$

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

This formula is on your exam paper.

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

Examples:

1. Find the Maclaurin series for the function $f(x) = e^{2x}$ up to and including the term x^4

$f(x) = e^{2x}$ $f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$ $f'(0) = 2e^0 = 2$

$f''(x) = 4e^{2x}$ $f''(0) = 4e^0 = 4$

$f'''(x) = 8e^{2x}$ $f'''(0) = 8e^0 = 8$

$f^{(iv)}(x) = 16e^{2x}$ $f^{(iv)}(0) = 16e^0 = 16$

$f(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \ldots$

$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \ldots = e^{2x}$

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

Examples:

2. Find the Maclaurin series for the function f(x) = cos2x up to and including the term x^4

$f(x) = \cos 2x$ $\quad f(0) = \cos 0 = 1$

$f'(x) = -2\sin 2x$ $\quad f'(0) = -2\sin 0 = 0$

$f''(x) = -4\cos 2x$ $\quad f''(0) = -4\cos 0 = -4$

$f'''(x) = 8\sin 2x$ $\quad f'''(0) = 8\sin 0 = 0$

$f^{(4)}(x) = 16\cos 2x$ $\quad f^{(4)}(0) = 16\cos 0 = 16$

$\cos 2x = 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 + \ldots\ldots$

$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 + \ldots$

Starter 30.1.2017

Given that $z = 1 - \sqrt{3}\,i$, write down $\bar{z}$ and express $\bar{z}^2$ in polar form. **4**

$\bar{z} = 1 + \sqrt{3}i$

$r = \sqrt{1 + \sqrt{3}^2}$

$= 2$

$\theta = \tan^{-1}(\sqrt{3})$

$= 60$

$\bar{z} = 2(\cos 60 + i\sin 60)$

$\bar{z}^2 = 4(\cos 120 + i\sin 120)$

Maclaurin Series

Examples:

3. Find the Maclaurin series for the function f(x) = ln(1 + 3x) up to and including the term x^4

$f(x) = \ln(1+3x)$ $\quad f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+3x} \cdot 3 = \frac{3}{1+3x} = 3(1+3x)^{-1}$ $\quad f'(0) = 3$

$f''(x) = -3(1+3x)^{-2}(3) = -9(1+3x)^{-2}$ $\quad f''(0) = -9$

$f'''(x) = 18(1+3x)^{-3}(3) = 54(1+3x)^{-3}$ $\quad f'''(0) = 54$

$f^{(4)}(x) = -162(1+3x)^{-4}(3) = -486(1+3x)^{-4}$ $\quad f^{(4)}(0) = -486$

$\ln(1+3x) = 0 + 3x - \frac{9}{2}x^2 + \frac{54}{3!}x^3 - \frac{486}{4!}x^4 + \ldots$

$= 3x - \frac{9x^2}{2} + 9x^3 - \frac{81}{4}x^4 + \ldots$

Maclaurin Series

Examples:

4. Find the Maclaurin series for the function f(x) = $\sqrt{1+4x}$ up to and including the term x^4

$f(x) = (1+4x)^{\frac{1}{2}}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+4x)^{-\frac{1}{2}}(4) = 2(1+4x)^{-\frac{1}{2}}$ $\quad f'(0) = 2$

$f''(x) = -1(1+4x)^{-3/2}(4) = -4(1+4x)^{-\frac{3}{2}}$ $\quad f''(0) = -4$

$f'''(x) = 6(1+4x)^{-5/2}(4) = 24(1+4x)^{-5/2}$ $\quad f'''(0) = 24$

$f^{(4)}(x) = -60(1+4x)^{-7/2}(4) = -240(1+4x)^{-7/2}$ $\quad f^{(4)}(0) = -240$

$\sqrt{1+4x} = 1 + 2x - \frac{4x^2}{2} + \frac{24x^3}{3!} - \frac{240x^4}{4!} + \ldots$

$= 1 + 2x - 2x^2 + 4x^3 - 10x^4 + \ldots$

Maclaurin Series - Composite Functions

Examples:

5. (a) Find the Maclaurin series for the function f(x) = $e^x \sin x$ up to and including the term x^4

(b) Hence obtain the Maclaurin series for the function g(x) = $e^{2x}\sin 2x$ up to and including the term in x^4

(a) $f(x) = e^x \sin x$ $\quad f(0) = 0$

$f'(x) = e^x(\cos x) + \sin x(e^x)$ $\quad f'(0) = 1$

$= e^x(\cos x + \sin x)$

$f''(x) = e^x(-\sin x + \cos x) + e^x(\cos x + \sin x)$ $\quad f''(0) = 2$

$= e^x(2\cos x) = 2e^x\cos x$ $\quad f'''(0) = 2$

$f'''(x) = 2e^x(-\sin x) + \cos x(2e^x)$ $\quad f^{(4)}(0) = 0$

$= 2e^x(\cos x - \sin x)$

$f^{(4)}(x) = 2e^x(-\sin x - \cos x) + (\cos x - \sin x)(2e^x)$

$= 2e^x(-2\sin x) = -4e^x\sin x$

$e^x \sin x = 0 + 1x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \frac{0x^4}{4!} + \ldots$

$= x + x^2 + \frac{1}{3}x^3 + \ldots$

(b) $e^{2x}\sin 2x$

Replace x with 2x

$= (2x) + (2x)^2 + \frac{(2x)^3}{3} + \ldots$

$= 2x + 4x^2 + \frac{8}{3}x^3 + \ldots$

Maclaurin Series - Composite Functions

Examples:

6. (a) Find the Maclaurin series for the function f(x) = $\sin^2 x$ up to and including the term x^4

(b) Hence obtain the Maclaurin series for the function $\cos^2 x$ up to and including the term in x^4

Maclaurin Series - Composite Functions

Examples:

6. (a) Find the Maclaurin series for the function $f(x) = \sin^2 x$ up to and including the term x^4

(b) Hence obtain the Maclaurin series for the function $\cos^2 x$ up to and including the term in x^4

(a) $f(x) = (\sin x)^2$ $\quad f(0) = 0$

$f'(x) = 2(\sin x)(\cos x) = 2\sin x\cos x$ $\quad f'(0) = 0$

$f''(x) = 2\sin x(-\sin x) + \cos x(2\cos x)$ $\quad f''(0) = 2$

$= -2\sin^2 x + 2\cos^2 x$ $\quad f'''(0) = 0$

$f'''(x) = -4\sin x\cos x - 4\cos x\sin x$ $\quad f^{(4)}(0) = -8$

$= -8\cos x\sin x$

$f^{(4)}(x) = -8\cos x(\cos x) + \sin x(8\sin x)$

$= 8\sin^2 x - 8\cos^2 x$

$\sin^2 x = 0 + 0x + \frac{2x^2}{2} + \frac{0x^3}{3!} - \frac{8x^4}{4!} + \ldots$

$= x^2 - \frac{x^4}{3} + \ldots$

(b) $\cos^2 x = 1 - \sin^2 x$

$= 1 - \left(x^2 - \frac{x^4}{3} + \ldots\right)$

$= 1 - x^2 + \frac{x^4}{3} + \ldots$

For certain functions we cannot find a MacLaurin series expansion as the derivative is undefined at x = 0.

1) $f(x) = \ln x$ $\quad f'(x) = \frac{1}{x}$ $\quad f'(0) = \frac{1}{0}$ which is undefined

2) $f(x) = \sqrt{x} = x^{1/2}$ $\quad f'(x) = \frac{1}{2}x^{-1/2} =$ $\quad f'(0) = \frac{1}{0}$ which is undefined.