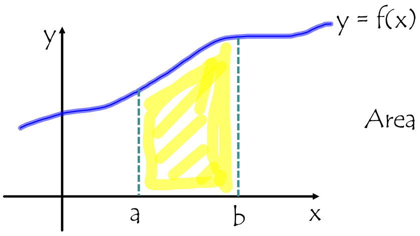
1.5 Applying algebraic and calculus skills to problems	
Sub-skill	**Description**
Applying differentiation to problems, in context	Apply differentiation to problems in context Apply differentiation to optimisation
Applying integration to problems, in context	Apply integration to volumes of revolution where the volume generated is by the rotation of the area under a single curve about the x and y axes Use calculus to determine corresponding connected integrals Apply integration to the evaluation of areas including integration with respect to y

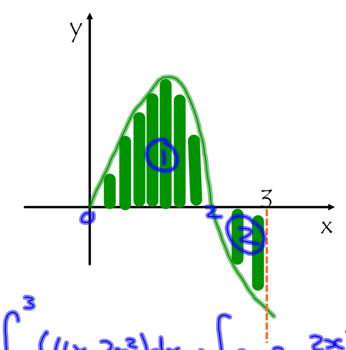
Applications of Integration - Area

Recap: Area between a curve and the x - axis



$$\text{Area} = \int_a^b y\,dx$$

Examples: 1. Find the area between the function y = 4x - 2x² and the x - axis



$4x-2x^2=0$

$2x(2-x)=0$

$x=0,\ x=2$

$$\int_0^2 (4x-2x^2)\,dx$$

$$=\left[\frac{4x^2}{2}-\frac{2x^3}{3}\right]_0^2 = \left[2x^2-\frac{2x^3}{3}\right]_0^2$$

$$=\left[8-\frac{16}{3}\right]-[0] = \frac{8}{3}\text{ units}^2$$

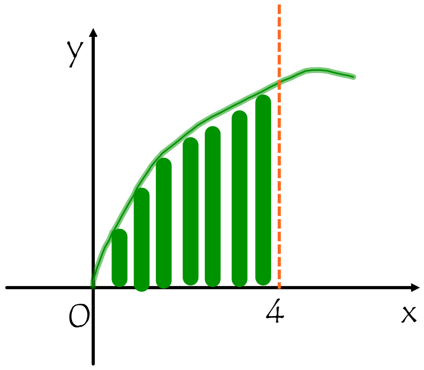
$$\int_2^3 (4x-2x^2)\,dx = \left[2x^2-\frac{2x^3}{3}\right]_2^3 = [18-18]-\left[8-\frac{16}{3}\right] = -\frac{8}{3}$$

$$=\frac{8}{3}\text{ units}^2$$

Total Area $= \frac{16}{3}$ units^2

Applications of Integration - Area

2. Calculate the area between f(x) and the x- axis where x = 0, x = 4



$$f(x) = \frac{4x}{\sqrt{x^2+2}}$$

$$\int_0^4 \frac{4x}{\sqrt{x^2+2}}$$

$u = x^2 + 2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$2du = 4x\,dx$

Change limits

$u = 4^2 + 2 = 18$

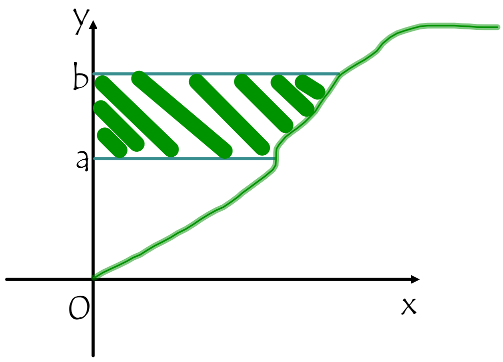
$u = 0^2 + 2 = 2$

$$= 2\int_2^{18} \frac{1}{u^{1/2}}\,du = 2\int_2^{18} u^{-\frac{1}{2}}\,du = \left[4u^{\frac{1}{2}}\right]_2^{18} = \left[4\sqrt{18}\right] - \left[4\sqrt{2}\right]$$

$$= 12\sqrt{2} - 4\sqrt{2} = \underline{8\sqrt{2}\text{ units}^2}$$

Applications of Integration - Area

Area between a curve and the y-axis



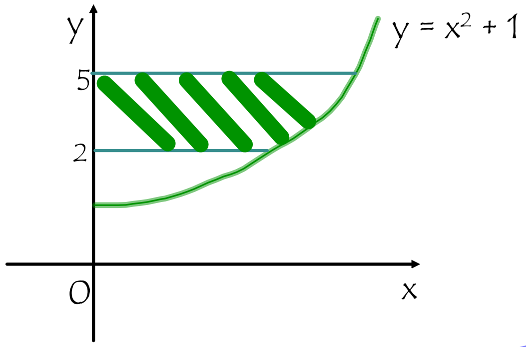
$$\text{Area} = \int_a^b x\,dy$$

To evaluate the integral, x must be expressed in terms of y.

Applications of Integration - Area

Examples:

1. Calculate the shaded area shown



$y = x^2 + 1$

$y - 1 = x^2$

$x = (y-1)^{\frac{1}{2}}$

$\int x\,dy = \int_2^5 (y-1)^{\frac{1}{2}}\,dy$

$= \left[\frac{(y-1)^{\frac{3}{2}}}{\frac{3}{2}(1)}\right]_2^5 = \left[\frac{2(\sqrt{y-1})^3}{3}\right]_2^5$

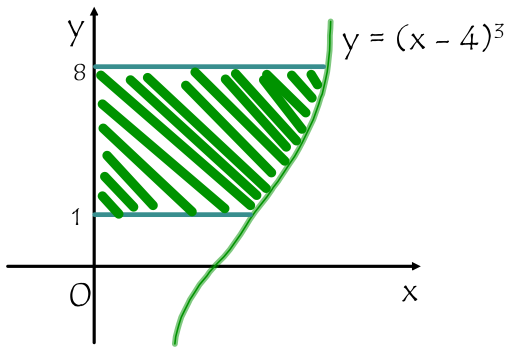
$= \left[\frac{2(\sqrt{4})^3}{3}\right] - \left[\frac{2(\sqrt{1})^3}{3}\right]$

$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$ units^2

Applications of Integration - Area

Examples:

2. Find the shaded area



$A = \int x\,dy$

$y = (x-4)^3$

$y^{\frac{1}{3}} = x - 4$

$\boxed{x = y^{\frac{1}{3}} + 4}$

$\int_1^8 \left(y^{\frac{1}{3}} + 4\right) dy$

$= \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}} + 4y\right]_1^8$

$= \left[\frac{3(\sqrt[3]{y})^4}{4} + 4y\right]_1^8 = \left[\frac{3(\sqrt[3]{8})^4}{4} + 4(8)\right] - \left[\frac{3(\sqrt[3]{1})^4}{4} + 4(1)\right]$

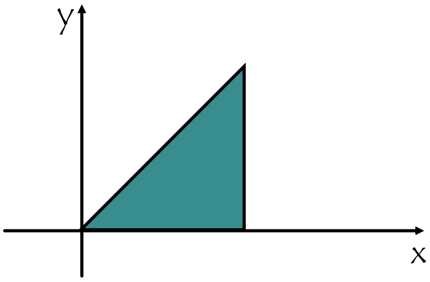
$= [12 + 32] - \left[\frac{3}{4} + 4\right]$

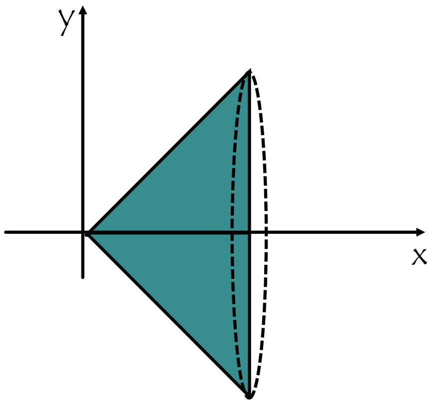
$= 44 - \frac{19}{4} = \frac{176}{4} - \frac{19}{4} = \frac{157}{4}$ units^2

Volumes of solids of revolutions

When a region in the xy plane is rotated through 360^0 about the x – axis, a solid of revolution is formed.

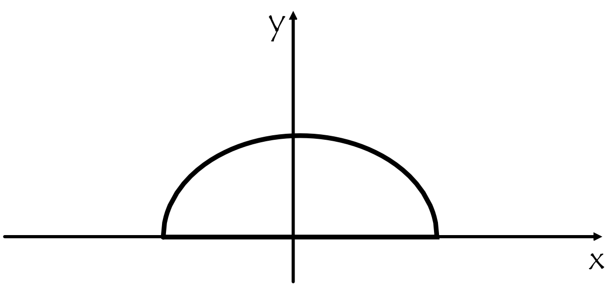
E.g. If the shaded triangle below is rotated 360^0 about the x – axis, a cone is formed.

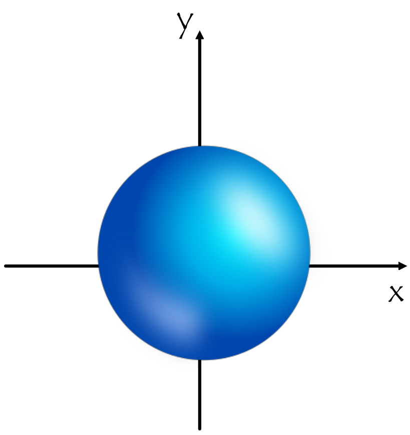




Volumes of solids of revolutions

When the shaded semi-circle is rotated through 360^0 about the x – axis, a sphere is formed.





Volumes of solids of revolutions

Area under a curve can be thought of as the sum of all the rectangles under a curve.

Volume of revolution can be thought of as the sum of cylindrical (approximately) discs where $V = \pi r^2 h$ with radius y and thickness dx.

$$V = \int_a^b \pi y^2 dx \quad \text{where } y = f(x)$$

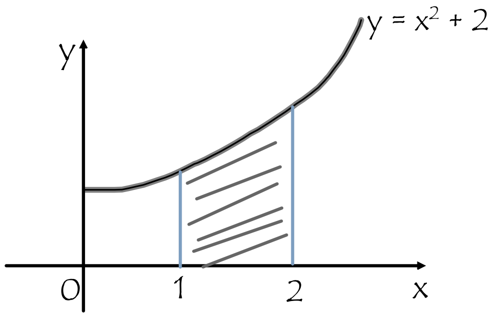
If we rotate about the y - axis, we get a similar result

$$V = \int_a^b \pi x^2 dy \quad \text{where } x = f(y)$$

Volumes of solids of revolutions

Examples:

1. Find the volume of the solid formed when the shaded region below is rotated 360^0 about the x - axis.



$$V = \int \pi y^2 dx$$

$$V = \pi \int_1^2 (x^2+2)^2 dx$$

$$V = \pi \int_1^2 (x^4 + 4x^2 + 4) dx$$

$$V = \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x\right]_1^2$$

$$V = \pi\left[\frac{32}{5} + \frac{32}{3} + 8\right] - \pi\left[\frac{1}{5} + \frac{4}{3} + 4\right]$$

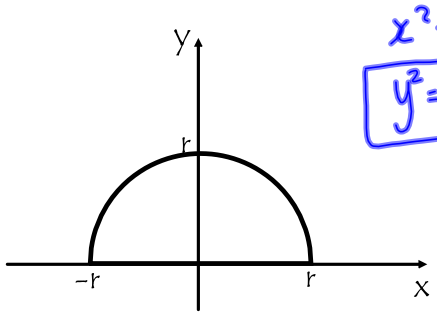
$$V = \pi\left[\frac{376}{15}\right] - \pi\left[\frac{83}{15}\right] = \frac{293\pi}{15} \text{ units}^3$$

Volumes of solids of revolutions

Examples:

2. Calculate the volume of the solid formed when the semi-circle with equation

$x^2 + y^2 = r^2$, $y \geq 0$ is given a full turn about the x - axis.



$$x^2+y^2=r^2$$
$$\boxed{y^2=r^2-x^2}$$

$$V=\pi\int y^2dx$$
$$V=\pi\int_{-r}^{r} r^2-x^2\,dx$$
$$V=\pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}$$

$$V=\pi\left[r^2(r)-\frac{r^3}{3}\right]-\pi\left[r^2(-r)-\frac{(-r)^3}{3}\right]$$
$$V=\pi\left[\frac{2r^3}{3}\right]-\pi\left[-\frac{2r^3}{3}\right]=\underline{\underline{\frac{4}{3}\pi r^3}}$$

Volumes of solids of revolutions

Past Paper Questions:

2004

11. A solid is formed by rotating the curve $y = e^{-2x}$ between $x = 0$ and $x = 1$ through 360° about the x-axis. Calculate the volume of the solid that is formed. **5**

$$V=\pi\int y^2dx$$
$$V=\pi\int_0^1 (e^{-2x})^2dx=\pi\int_0^1 e^{-4x}dx$$
$$=\pi\left[\frac{e^{-4x}}{-4}\right]_0^1=\pi\left[\frac{e^{-4}}{-4}\right]-\pi\left[\frac{e^0}{-4}\right]=\frac{-\pi}{4e^4}+\frac{\pi}{4}$$
$$=\underline{\underline{\frac{\pi}{4}\left(1-\frac{1}{e^4}\right)}}$$

Q10. 2007
Q10. 2014
Q16. 2017

Volumes of solids of revolutions

Past Paper Questions:

2007

10. Use the substitution $u = 1 + x^2$ to obtain $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$. **5**

A solid is formed by rotating the curve $y = \frac{x^{3/2}}{(1+x^2)^2}$ between $x = 0$ and $x = 1$ through 360° about the x-axis. Write down the volume of this solid. **1**

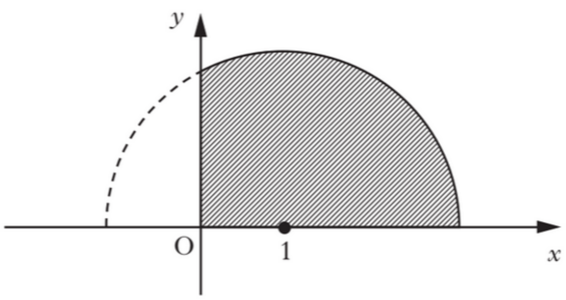
Volumes of solids of revolutions

Past Paper Questions:

2007

10. A semi-circle with centre (1, 0) and radius 2, lies on the x-axis as shown.

Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis. **5**



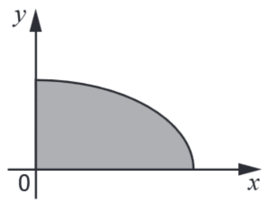
Volumes of solids of revolutions

Past Paper Questions:

2017

16. On a suitable domain, a curve is defined by the equation $4x^2 + 9y^2 = 36$.

A section of the curve in the first quadrant, illustrated in the diagram below, is rotated 360° about the **y-axis**.



Calculate the exact value of the volume generated. **5**