Differential Equations

A differential equation is an equation involving one or more derivatives.

$(x^2-1)\frac{dy}{dx}+2xy=x$ is a first order differential equation as it contains the first derivative only.

$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$ is a second order differential equation as it contains the second derivative.

Differential Equations

If we have a function y = f(x), we can verify that the function satisfies a particular differential equation.

For example:

Verify that $y=Ae^{5x}+Be^{-2x}$ where A, B are constants

satisfies the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}-10y=0$

$y=Ae^{5x}+Be^{-2x}$

$\frac{dy}{dx}=5Ae^{5x}-2Be^{-2x}$

$\frac{d^2y}{dx^2}=25Ae^{5x}+4Be^{-2x}$

$25Ae^{5x}+4Be^{-2x}-3(5Ae^{5x}-2Be^{-2x})-10(Ae^{5x}+Be^{-2x})$

$=25Ae^{5x}+4Be^{-2x}-15Ae^{5x}+6Be^{-2x}-10Ae^{5x}-10Be^{-2x}$

$10Ae^{5x}-10Ae^{5x}+10Be^{-2x}-10Be^{-2x}$

$=0$ $\Rightarrow$ satisfies the equation

General solution of differential equations

If a first order differential equation can be written in the form

$$\boxed{f(y)dy=g(x)dx}$$

where f(y) is a function of y only and g(x) is a function of x only, we say that we have separated the variables x and y.

The general solution of the differential equation is then found by integrating each side with respect to the appropriate variable.

$$\boxed{\int f(y)dy=\int g(x)dx}$$

General solution of differential equations

1. Find the general solution of the differential equation

$\frac{dy}{dx}=x^3-x^2$

$\int 1\,dy=\int(x^3-x^2)dx$

$y=\frac{x^4}{4}-\frac{x^3}{3}+C$

General solution of differential equations

2. Find the general solution of the differential equation

$\frac{dy}{dx}=\frac{x^2}{y}$

$\int y\,dy=\int x^2\,dx$

$\frac{y^2}{2}=\frac{x^3}{3}+C$

$3y^2=2x^3+C$

$y^2=\frac{2}{3}x^3+C$ Total constant

$y=(\frac{2}{3}x^3+C)^{\frac{1}{2}}$

General solution of differential equations

3. Find the general solution of the differential equation

$e^{4y}\frac{dy}{dx}-x=2$

$e^{4y}\frac{dy}{dx}=2+x$

$e^{4y}dy=(2+x)dx$

$\int e^{4y}dy=\int(2+x)dx$

$\frac{1}{4}e^{4y}=2x+\frac{x^2}{2}+C$

$e^{4y}=8x+2x^2+C$

Take the $\log_e$ of both sides

$4y=\ln|8x+2x^2+C|$

$y=\frac{1}{4}\ln|8x+2x^2+C|$

General solution of differential equations

4. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{16+4x^2}$$

$1\,dy = \frac{1}{16+4x^2}\,dx$

$\int 1\,dy = \int \frac{1}{4(4+x^2)}\,dx$

$\int 1\,dy = \frac{1}{4}\int \frac{1}{2^2+x^2}\,dx$

$y = \frac{1}{4}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right] + C$

$y = \frac{1}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$

$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

General solution of differential equations

5. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{2}{\sin y}$$

$\sin y\,dy = 2\,dx$

$\int \sin y\,dy = \int 2\,dx$

$-\cos y = 2x + C$

$\cos y = -2x + C$

$y = \cos^{-1}(-2x + C)$

General solution of differential equations

6. Find the general solution of the differential equation

$$\frac{dy}{dx} = 2x\sqrt{9-y^2}$$

$dy = 2x\sqrt{9-y^2}\,dx$

$\int \frac{1}{\sqrt{9-y^2}}\,dy = \int 2x\,dx$

$\sin^{-1}\left(\frac{y}{3}\right) = x^2 + C$

$\frac{y}{3} = \sin(x^2 + C)$

$y = 3\sin(x^2 + C)$

General solution of differential equations

7. Find the general solution of the differential equation

$$(1+x^2)y^2\frac{dy}{dx} = 1$$

$\int y^2\,dy = \int \frac{1}{1+x^2}\,dx$

$\frac{y^3}{3} = \tan^{-1}(x) + C$

$y^3 = 3\tan^{-1}(x) + C$

$y = (3\tan^{-1}(x) + C)^{\frac{1}{3}}$

General solution of differential equations

We can use properties of logs and exponents to help us simplify general solutions.

Examples:

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = 4y$$

$dy = 4y\,dx$

$\int \frac{1}{y}\,dy = \int 4\,dx$

$\ln|y| = 4x + C$

$e^{\wedge} \qquad e^{\wedge}$

$y = e^{4x+C}$

$y = e^{4x}.e^{C}$

let $A = e^{C}$

$y = Ae^{4x}$

General solution of differential equations

2. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y+2}{x+1} \longrightarrow (x+1)dy = (y+2)dx \longrightarrow \frac{1}{y+2}\,dy = \frac{1}{x+1}\,dx$$

$\int \frac{1}{y+2}\,dy = \int \frac{1}{x+1}\,dx$

$\ln|y+2| = \ln|x+1| + C$

$e^{\wedge} \qquad e^{\wedge}$

$y+2 = (x+1).e^{C}$

$y = (x+1).e^{C} - 2$

$e^{\ln|x+1|+C} = e^{\ln|x+1|}.e^{C} = (x+1)e^{C}$

let $A = e^{C}$

$y = A(x+1) - 2$

General solution of differential equations

3. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{2x-1}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{2x-1}\,dx$$

$$\ln|y| = \tfrac{1}{2}\ln|2x-1| + C$$

$$\ln|y| = \ln(2x-1)^{\frac{1}{2}} + C$$

$$e^{\wedge} \qquad e^{\wedge}$$

$$y = e^{c}.(2x-1)^{\frac{1}{2}} \qquad \text{let } A = e^{c}$$

$$y = A(2x-1)^{\frac{1}{2}}$$

Differential equations with partial fractions

Examples:

1. (a) Express $\frac{2x+3}{x(x+1)}$ in partial fractions

$$\frac{A}{x} + \frac{B}{x+1} = \frac{2x+3}{x(x+1)}$$

$$A(x+1) + Bx = 2x+3$$

let $x = 0$: $A = 3$; let $x = -1$: $-B = 1$, $B = -1$

$$\frac{3}{x} - \frac{1}{x+1}$$

(b) Hence find the general solution of the differential equation

$$x(x+1)\frac{dy}{dx} = y(2x+3)$$

expressing y explicitly in terms of x

$$\int \frac{1}{y}\,dy = \int \frac{2x+3}{x(x+1)}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{3}{x} - \frac{1}{x+1}\,dx$$

$$\ln|y| = 3\ln|x| - \ln|x+1| + C$$

$$\ln|y| = \ln\left|\frac{x^3}{x+1}\right| + C$$

$$e^{\wedge} \qquad e^{\wedge}$$

$$y = \frac{x^3}{x+1}.e^{c} \qquad \text{let } A = e^{c}$$

$$y = \frac{Ax^3}{x+1}$$

Differential equations with partial fractions

2. (a) Express $\frac{x+1}{x(2x+1)}$ in partial fractions.

(b) Hence find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y(x+1)}{x(2x+1)}$$

expressing y explicitly in terms of x

(a) $$\frac{x+1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1} = \frac{1}{x} - \frac{1}{2x+1}$$

$$A(2x+1) + B(x) = x+1$$

let $x = 0$: $A = 1$; let $x = -\frac{1}{2}$: $-\frac{1}{2}B = \frac{1}{2}$, $B = -1$

(b) $$\frac{dy}{dx} = \frac{y(x+1)}{x(2x+1)}$$

$$\frac{1}{y}\,dy = \frac{x+1}{x(2x+1)}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{x+1}{x(2x+1)}\,dx$$

$$\ln|y| = \int \frac{1}{x} - \frac{1}{2x+1}\,dx$$

$$\ln|y| = \ln|x| - \tfrac{1}{2}\ln|2x+1| + C$$

$$\ln|y| = \ln\frac{x}{\sqrt{2x+1}} + C$$

$$e^{\wedge} \qquad \underbrace{\qquad}_{e^{\wedge}}$$

$$e^{\ln\frac{x}{\sqrt{2x+1}} + c} = e^{\ln\frac{x}{\sqrt{2x+1}}}.e^{c}$$

$$y = e^{c}\left(\frac{x}{\sqrt{2x+1}}\right) \qquad \text{let } A = e^{c}$$

$$y = \frac{Ax}{\sqrt{2x+1}}$$

Differential equations with partial fractions

3. Find the general solution of the differential equation

$$x(x+1)\frac{dy}{dx} = y(3x+1)$$

expressing y explicitly in terms of x

$$x(x+1)\,dy = y(3x+1)\,dx$$

$$\frac{1}{y}\,dy = \underbrace{\frac{3x+1}{x(x+1)}\,dx}$$

Turn into partial fractions

$$\frac{A}{x} + \frac{B}{x+1} \Rightarrow A(x+1) + B(x) = 3x+1$$

let $x = 0$: $A = 1$; let $x = -1$: $-B = -2$, $B = 2$

$$\frac{1}{y}\,dy = \frac{1}{x} + \frac{2}{x+1}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x} + \frac{2}{x+1}\,dx$$

$$\Rightarrow \ln|y| = \ln|x| + 2\ln|x+1| + C$$

$$\ln|y| = \ln x(x+1)^2 + C$$

$$e^{\wedge} \qquad e^{\wedge}$$

$$y = x(x+1)^2.e^{c} \qquad \text{let } e^{c} = A$$

$$y = Ax(x+1)^2$$

Particular Solutions of differential equations

If we are given initial conditions for x and y, we can find the particular solution of a differential equation.

Examples:

1. Find the particular solution of the differential equation

$$y^2\frac{dy}{dx} = 4x^2 + 1$$

given that y = 2 when x = 1, expressing y in terms of x.

$$y^2\,dy = (4x^2+1)\,dx$$

$$\int y^2\,dy = \int (4x^2+1)\,dx$$

$$\Rightarrow \frac{y^3}{3} = \frac{4x^3}{3} + x + C$$

$$\Rightarrow y^3 = 4x^3 + 3x + C$$

$y = 2$ when $x = 1$

$$2^3 = 4 + 3 + C$$

$$8 = 7 + C$$

$$C = 1$$

$$y = (4x^3 + 3x + 1)^{\frac{1}{3}}$$

Particular Solutions of differential equations

2. Find the particular solution of the differential equation

$$\frac{dy}{dx} = e^{3x+2} \qquad 1\,dy = e^{3x+2}\,dx$$

given that y = e^2 when x = 0

$$\int 1\,dy = \int e^{3x+2}\,dx$$

$$\Rightarrow y = \tfrac{1}{3}e^{3x+2} + C$$

$$\boxed{y = e^2 \quad x = 0}$$

$$e^2 = \tfrac{1}{3}e^2 + C$$

$$\tfrac{2}{3}e^2 = C$$

$$y = \tfrac{1}{3}e^{3x+2} + \tfrac{2}{3}e^2$$

Particular Solutions of differential equations

3. Find the particular solution of the differential equation

$$\frac{dy}{dx} = 2xy^2$$

given that $y = -\frac{3}{10}$ when $x = -2$

$\frac{1}{y^2}\,dy = 2x\,dx$

$y^{-2}\,dy = 2x\,dx$

$\int y^{-2}\,dy = \int 2x\,dx$

$-\frac{1}{y} = x^2 + C$

$\boxed{y = -\frac{3}{10} \quad x = -2}$

$\frac{10}{3} = 4 + C$

$\boxed{-\frac{2}{3} = C}$

$-1 = y(x^2 + c)$

$y = \frac{-1}{x^2 - \frac{2}{3}}$

$\Rightarrow y = \frac{-3}{3x^2 - 2}$

Particular Solutions of differential equations

4. Find the particular solution of the differential equation

$$(x+4)\frac{dy}{dx} = y - 1$$

given that y = 7 when x = −1

$(x+4)\,dy = (y-1)\,dx$

$\int \frac{1}{y-1}\,dy = \int \frac{1}{x+4}\,dx$

$= \ln|y-1| = \ln|x+4| + C$

$e^\wedge \quad e^\wedge$

$y - 1 = (x+4).e^c \quad$ let $A = e^c$

$y - 1 = A(x+4)$

$y = A(x+4) + 1$

$\boxed{\begin{matrix} y = 7 \\ x = -1 \end{matrix}}$

$7 = A(-1+4) + 1$

$7 = 3A + 1$

$3A = 6$

$A = 2$

$y = 2(x+4) + 1$

$y = 2x + 9$

Particular Solutions of differential equations

5. (a) Express $\frac{2}{(y+1)(y+3)}$ in partial fractions

(b) Hence find the particular solution of the differential equation

$$2x\frac{dy}{dx} = (y+1)(y+3)$$

given that $y = -\frac{5}{3}$ when x = −1

(a) $\frac{2}{(y+1)(y+3)} = \frac{A}{y+1} + \frac{B}{y+3}$

$A(y+3) + B(y+1) = 2$

let $y = -1$ let $y = -3$

$2A = 2 \quad -2B = 2$

$A = 1 \quad B = -1$

$\frac{1}{y+1} - \frac{1}{y+3}$

(b) $x\,dy = \frac{(y+1)(y+3)}{2}\,dx$

$\int \frac{2}{(y+1)(y+3)}\,dy = \int \frac{1}{x}\,dx$

$\int \frac{1}{y+1} - \frac{1}{y+3}\,dy = \int \frac{1}{2}\,dx$

$\ln|y+1| - \ln|y+3| = \ln|x| + c$

$\ln\left|\frac{y+1}{y+3}\right| = \ln|x| + c$

$e^\wedge \quad e^\wedge$

$\frac{y+1}{y+3} = x.e^c \quad$ let $A = e^c$

$\frac{y+1}{y+3} = Ax \quad y = -\frac{5}{3},\ x = -1$

$\frac{-\frac{5}{3}+1}{-\frac{5}{3}+3} = -A \quad \frac{-5+3}{-5+9} = -A$

$A = \frac{1}{2} \quad \frac{-2}{4} = -A$

$\frac{1}{2}x = \frac{y+1}{y+3}$

$x = \frac{2y+2}{y+3}$

$x(y+3) = 2y + 2$

$xy + 3x = 2y + 2$

$xy - 2y = 2 - 3x$

$y(x-2) = 2 - 3x$

$y = \frac{2-3x}{x-2}$

Differential Equations as Mathematical Models

Examples:

1. The number of strands of bacteria x present in a culture after t days of growth is assumed to be increasing at a rate that is proportional to the number of strands present.

(a) Write down a differential equation which represents this and find the general solution for x in terms of t.

(b) Given that there are 326 strands initially present and that the number of strands observed after 4 days is 1833, estimate the number of strands likely to be present after 1 week.

$\frac{dx}{dt}$ is proportional to x

(a) so let $\frac{dx}{dt} = kx$ where k is a constant

$dx = kx\,dt$

$\int \frac{1}{x}\,dx = \int k\,dt$

$\ln|x| = kt + c$

$e^\wedge \quad e^\wedge$

$x = e^{kt}.e^c \quad$ let $e^c = A$

$x = Ae^{kt}$

When $t = 7$

$x = 326e^{0.4317(7)}$

$x \approx 6692.9$

(b) $x = 326$ at $t = 0$

$326 = Ae^0$

$326 = A$

$1833 = 326e^{4k}$

$\frac{1833}{326} = e^{4k}$

$\ln\left(\frac{1833}{326}\right) = 4k$

$k = \frac{1}{4}\ln\left(\frac{1833}{326}\right) \approx 0.4317$

Particular Solutions of differential equations

Past Paper Question:

5 marks

2017

Solve $\frac{dy}{dx} = e^{2x}(1+y^2)$ given that when x = 0, y = 1. Express y in terms of x

$dy = e^{2x}(1+y^2)\,dx$

$\int \frac{1}{1+y^2}\,dy = \int e^{2x}\,dx$

$\tan^{-1}(y) = \frac{1}{2}e^{2x} + C$

$y = \tan\left(\frac{1}{2}e^{2x} + C\right)$

$\begin{matrix} x & y \\ (0, & 1) \end{matrix}$

$1 = \tan\left(\frac{1}{2} + C\right)$

$\tan^{-1} 1 = \frac{1}{2} + C$

$\frac{\pi}{4} = \frac{1}{2} + C$

$C = \frac{\pi}{4} - \frac{1}{2}$

$y = \tan\left(\frac{1}{2}e^{2x} + \frac{\pi}{4} - \frac{1}{2}\right)$

Differential Equations as Mathematical Models

Examples:

2. The mass m (grams) of a radioactive substance at time t (years) decreases at a rate which is proportional to the mass at that time. The half-life of a radioactive substance is the time taken for half of the mass to decay. The original mass ($t=0$) of a radioactive substance is 700grams and after 8 years it has decayed to 550grams. Calculate the half-life of this radioactive substance.

(proportion constant negative because mass decreasing)

$\frac{dm}{dt} = -km$

$dm = -km\,dt$

$\int \frac{1}{m}\,dm = \int -k\,dt$

$\ln|m| = -kt + c$

$e^\wedge \quad e^\wedge$

$m = e^{-kt}.e^c \quad$ Let $A = e^c$

$m = Ae^{-kt}$

$t = 0,\ m = 700$

$700 = Ae^0$

$A = 700$

$t = 8,\ m = 550,\ A = 700$

$550 = 700e^{-8k}$

$\frac{550}{700} = e^{-8k}$

$\ln\left(\frac{550}{700}\right) = -8k$

$k = \frac{\ln\left(\frac{550}{700}\right)}{-8} = 0.030145257$

$t = ?\quad m = 350$

$350 = 700e^{-0.030145257t}$

$\frac{1}{2} = e^{-0.03\ldots t}$

$\ln\frac{1}{2} = -0.03\ldots t$

$t = \frac{\ln\frac{1}{2}}{-0.03\ldots} = 22.99\ldots \approx 23$ years

Differential Equations as Mathematical Models

Examples:

3. According to Newton's Law of Cooling, the rate at which the object cools is proportional to the difference in temperature between the object and the surrounding medium. When an object cools surrounded by air at a temperature of 75°C, the cooling of the object is governed by a differential equation of the form

$$\frac{dT}{dt} = -k(T-75)$$

where T°C is the temperature of the object after t hours of cooling and k is a constant.

(a) Find the general solution of the differential equation, expressing T as a function of t.

(b) Given that a certain object cools from 125°C to 100°C in half an hour surrounded by air at a temperature of 75°C, find

(i) the temperature of this object at the end of another half-hour

(ii) the time taken for the temperature of this object to fall to 80°C

(a) $\frac{dT}{dt} = -k(T-75)$

$dT = -k(T-75)\,dt$

$\int \frac{1}{T-75}\,dT = \int -k\,dt$

$\ln|T-75| = -kt + c$

$e^{\wedge}$ $e^{\wedge}$

$T-75 = e^{-kt}.e^{c}$ let $e^c = A$

$T-75 = Ae^{-kt}$

$T = Ae^{-kt} + 75$

(b) $t=0$, $T = 125°C$

$125 = Ae^0 + 75$

$125 = A + 75$

$A = 50$

$t = 0.5$ $T = 100°C$

$100 = 50e^{-0.5k} + 75$

$25 = 50e^{-0.5k}$

$\frac{1}{2} = e^{-0.5k}$

$\ln\frac{1}{2} = -0.5k$

$k = \frac{\ln 0.5}{-0.5} = 1.386294361$

(i) $t = 1$ $T = ?$

$T = 50e^{-1.386294361} + 75$

$T = 87.5°C$ after 1 hour

(ii) $T = 80°C$ $t = ?$

$80 = 50e^{-1.386294361t} + 75$

$5 = 50e^{-1.386294361t}$

$0.1 = e^{-1.386294361t}$

$\ln 0.1 = -1.386.....t$

$t = \frac{\ln 0.1}{-1.386.....} \approx 1.66$ hours $\approx$ 1hour 40mins

Differential Equations as Mathematical Models

2016 Past Paper

A beaker of liquid was placed in a fridge.

The rate of cooling is given by

$$\frac{dT}{dt} = -k(T - T_F), \quad k > 0,$$

where T_F is the constant temperature in the fridge and T is the temperature of the liquid at time t.

- The constant temperature in the fridge is 4 °C.
- When first placed in the fridge, the temperature of the liquid was 25 °C.
- At 12 noon, the temperature of the liquid was 9·8 °C.
- At 12:15 pm, the temperature of the liquid had dropped to 6·5 °C.

At what time, to the nearest minute, was the liquid placed in the fridge? 9

- $T_F = 4 \Rightarrow \frac{dT}{dt} = -k(T-4)$

$dT = -k(T-4)\,dt$

$\int \frac{1}{T-4}\,dT = \int -k\,dt$

$\ln|T-4| = -kt + c$

$e^{\wedge}$ $e^{\wedge}$

$T-4 = e^{-kt}.e^{c}$ $\boxed{\text{let } e^c = A}$

$T-4 = Ae^{-kt}$

$T = Ae^{-kt} + 4$

- $t=0$ $T = 25$

$25 = Ae^0 + 4$

$25 = A + 4$

$A = 21°C$

- $t = t$ $T = 9.8°C$

$9.8 = 21e^{-kt} + 4$

$5.8 = 21e^{-kt}$

$\frac{5.8}{21} = e^{-kt}$

- $t = t+15$ $T = 6.5°C$

$6.5 = 21e^{-k(t+15)} + 4$

$\frac{2.5}{21} = e^{-kt-15k}$

$\frac{2.5}{21} = e^{-kt}.e^{-15k}$

- $\frac{\frac{2.5}{21}}{\frac{5.8}{21}} = e^{-15k} \Rightarrow \frac{2.5}{5.8} = e^{-15k} \Rightarrow \ln\left(\frac{2.5}{5.8}\right) = -15k$ $\quad k = \frac{\ln\left(\frac{2.5}{5.8}\right)}{-15}$

$k = 0.056104479$

- $\frac{5.8}{21} = e^{-kt}$ $\quad \ln\left(\frac{5.8}{21}\right) = -0.056104479t$

$t = 22.9\text{mins} \approx 23\text{mins}$

- It was placed in the fridge at 12pm − 23mins = 11:37am

Further Differential Equations

So far we have only been working with non-linear first order differential equations that are separable.

What about this type of equation?

$$\frac{dy}{dx} + 2xy = 6e^{2x}$$

Further Differential Equations

A first order differential equation where the variables cannot be separated is known as a first order linear differential equation and is generally of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

We need to perform some manipulation to get the equation into a form that we can work with. The method we use is called the integrating factor method.

$I(x) = e^{\int P(x)dx}$ is known as the integrating factor.

Further Differential Equations

Examples:

1. Find the general solution of the differential equation

$$\frac{dy}{dx} + y = 6e^{2x}$$

$\frac{dy}{dx} + P(x)y = Q(x)$

$I(x) = e^{\int P(x)dx}$

$P(x) = 1$ $\quad I(x) = e^{\int 1dx} = e^x$

Multiply through by $I(x)$

$e^x \, dy/dx + e^x y = 6e^{3x}$

from product rule $\longrightarrow$ $\frac{d}{dx}(e^x y) = 6e^{3x}$

$\int \frac{d}{dx}(e^x y) = \int 6e^{3x}\,dx$

$e^x y = 2e^{3x} + c$

$y = 2e^{2x} + \frac{c}{e^x}$

Further Differential Equations

Examples:

2. Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = e^{3x}$$

$\frac{dy}{dx} + P(x)y = Q(x)$

$I(x) = e^{\int P(x)dx}$

$P(x) = 2$ $\quad I(x) = e^{\int 2dx} = e^{2x}$

$\underbrace{e^{2x}\frac{dy}{dx} + 2e^{2x}y}$ $= e^{5x}$

$\frac{d}{dx}(e^{2x}y) = e^{5x}$

$e^{2x}y = \int e^{5x}\,dx$

$e^{2x}y = \frac{1}{5}e^{5x} + c$

$y = \frac{1}{5}e^{3x} + \frac{c}{e^{2x}}$

Further Differential Equations

Examples:

3. Find the general solution of the differential equation

$$\frac{dy}{dx}+\frac{y}{x}=x$$

$$\boxed{\frac{dy}{dx}+P(x)y=Q(x)}$$

$$\boxed{I(x)=e^{\int P(x)dx}}$$

$P(x)=\frac{1}{x}$ $\quad I(x)=e^{\int \frac{1}{x}dx}=e^{\ln x}=x$

$$\boxed{e^{\ln x}=x}$$

$$x\frac{dy}{dx}+y=x^2$$

$$\frac{d}{dx}(xy)=x^2$$

$$xy=\int x^2dx$$

$$xy=\frac{x^3}{3}+C$$

$$y=\frac{x^2}{3}+\frac{C}{x}$$

Further Differential Equations

Examples:

4. Find the general solution of the differential equation

$$\frac{dy}{dx}=\frac{xy+4}{x^2}$$

$$\boxed{\frac{dy}{dx}+P(x)y=Q(x)}$$

$$\boxed{I(x)=e^{\int P(x)dx}}$$

$$\frac{dy}{dx}=\frac{xy}{x^2}+\frac{4}{x^2}$$

$$\frac{dy}{dx}=\frac{y}{x}+\frac{4}{x^2}$$

$$\frac{dy}{dx}-\frac{1}{x}y=\frac{4}{x^2}$$

$P(x)=-\frac{1}{x}$ $\quad I(x)=e^{\int -\frac{1}{x}dx}=e^{\ln x^{-1}}=\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=\frac{4}{x^3}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right)=\frac{4}{x^3}$$

$$\frac{1}{x}y=\int 4x^{-3}dx$$

$$\frac{1}{x}y=-2x^{-2}+C$$

$$y=\frac{-2}{x}+Cx$$

Further Differential Equations

Examples:

5. Find the general solution of the differential equation

$$x\frac{dy}{dx}-2y=x^3\sin x$$

$$\boxed{\frac{dy}{dx}+P(x)y=Q(x)}$$

$$\boxed{I(x)=e^{\int P(x)dx}}$$

$$\frac{dy}{dx}-\frac{2y}{x}=x^2\sin x$$

$P(x)=-2x^{-1}$ $\quad I(x)=e^{\int -2x^{-1}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}$

$$x^{-2}\frac{dy}{dx}-\frac{2y}{x^3}=\sin x$$

$$\frac{d}{dx}(x^{-2}y)=\sin x$$

$$x^{-2}y=\int \sin x\,dx$$

$$x^{-2}y=-\cos x+C$$

$\frac{1}{x^2}y=-\cos x+C$

$$y=-x^2\cos x+Cx^2$$

Further Differential Equations

Examples:

6. Find the general solution of the differential equation

$$\frac{dy}{dx}+2xy=5x$$

$$\boxed{\frac{dy}{dx}+P(x)y=Q(x)}$$

$$\boxed{I(x)=e^{\int P(x)dx}}$$

$P(x)=2x$ $\quad I(x)=e^{\int 2x\,dx}=e^{x^2}$

$$e^{x^2}\frac{dy}{dx}+2e^{x^2}xy=5e^{x^2}x$$

$$\frac{d}{dx}\left(e^{x^2}y\right)=5e^{x^2}x$$

$$e^{x^2}y=\int 5e^{x^2}x\,dx=5\int e^{x^2}x\,dx$$

let $u=x^2$

$\frac{du}{dx}=2x$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$$\frac{5}{2}\int e^u\,du$$

$$e^{x^2}y=\frac{5}{2}e^{x^2}+C$$

$$y=\frac{5}{2}+\frac{C}{e^{x^2}}$$

Further Differential Equations

Examples:

7. Find the general solution of the differential equation

$$\cos x\frac{dy}{dx}+y\sin x=\cos^2 x$$

$$\boxed{\frac{dy}{dx}+P(x)y=Q(x)}$$

$$\boxed{I(x)=e^{\int P(x)dx}}$$

$$\frac{dy}{dx}+y\tan x=\cos x$$

$P(x)=\tan x$ $\quad I(x)=e^{\int \frac{\sin x}{\cos x}}=e^{-\ln(\cos x)}=\frac{1}{\cos x}=\sec x$

$\int \frac{\sin x}{\cos x}dx$ $\quad$ let $u=\cos x$

$\int -\frac{1}{u}du$ $\quad \frac{du}{dx}=-\sin x$

$=-\ln|\cos x|$ $\quad du=-\sin x\,dx$

$-du=\sin x\,dx$

$$\sec x\frac{dy}{dx}+y\tan x\sec x=\cos x\sec x$$

$$\frac{d}{dx}(y\sec x)=1$$

$$y\sec x=\int 1\,dx$$

$$y\sec x=x+C$$

$$y=\frac{x}{\sec x}+\frac{C}{\sec x}$$

$\sec x=\frac{1}{\cos x}$

$\frac{1}{\sec x}=\cos x$

$$y=x\cos x+C\cos x$$

Further Differential Equations - Particular Solutions

1. Find the particular solution for the differential equation

$$x\frac{dy}{dx}+3y=5x^2$$

$$\boxed{\frac{dy}{dx}+P(x)y=Q(x)}$$

$$\boxed{I(x)=e^{\int P(x)dx}}$$

given that y = 3 when x = 1

$$\frac{dy}{dx}+\frac{3}{x}y=5x$$

$P(x)=\frac{3}{x}$ $\quad I(x)=e^{\int 3x^{-1}dx}=e^{3\ln x}=x^3$

$$x^3\frac{dy}{dx}+3x^2y=5x^4$$

$$\frac{d}{dx}(x^3y)=5x^4$$

$$x^3y=\int 5x^4dx$$

$$x^3y=x^5+C$$

$$y=x^2+\frac{C}{x^3}$$

$y=3$ when $x=1$

$$3=1+C$$

$$C=2$$

$$y=x^2+\frac{2}{x^3}$$

Further Differential Equations - Particular Solutions

2. Find the particular solution for the differential equation

$$x\frac{dy}{dx} - y = x^2$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

given that x = 1 when y = 0

$$I(x) = e^{\int P(x)dx}$$

$\frac{dy}{dx} - \frac{1}{x}y = x$

$P(x) = -\frac{1}{x}$ $\quad I(x) = e^{\int -\frac{1}{x}dx} = e^{\ln x^{-1}} = \frac{1}{x}$

$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 1$

$\frac{d}{dx}\left(\frac{1}{x}y\right) = 1$

$\frac{1}{x}y = \int 1dx$

$\frac{1}{x}y = x + c$

$y = x^2 + cx$

When to use integration by parts:

- Always try substitution first.
- If neither factor is related to the derivative of the other, use integration by parts.

Second Order Differential Equations

These are equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b and c are constants, a ≠ 0 and f(x) is a function of x only.

If f(x) = 0, the differential equation is said to be homogeneous.

If f(x) ≠ 0, the differential equation is said to be non-homogeneous.

Homogeneous 2nd order differential equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

As these equations will involve a 2nd derivative, the general solution will have 2 arbitary constants.

We find the general solution by considering a solution of the form $y = e^{mx}$

$y = e^{mx}$ $\quad \frac{dy}{dx} = me^{mx}$ $\quad \frac{d^2y}{dx^2} = m^2e^{mx}$

$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$ $\quad \div e^{mx}$ on both sides

$am^2 + bm + c = 0$

Homogeneous 2nd order differential equations

$am^2 + bm + c = 0$ is known as the auxilliary equation and can be factorised to solve for m.

The nature of the roots can be found by using $b^2 - 4ac$.

$b^2 - 4ac > 0$	2 real roots	$y = Ae^{m_1x} + Be^{m_2x}$
$b^2 - 4ac = 0$	2 equal roots	$y = e^{mx}(A + Bx)$
$b^2 - 4ac < 0$	2 complex roots	$y = e^{px}(A\cos qx + B\sin qx)$

$m = p \pm qi$

https://www.khanacademy.org/math/differential-equations

For further reading on proofs!

Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

1. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$

$m^2 + 2m - 8 = 0$

$(m+4)(m-2) = 0$

$m = -4$, $m = 2$

$y = e^{mx}$

$y = e^{-4x}$ $\quad y = e^{2x}$

$y = Ae^{-4x}$ $\quad y = Be^{2x}$

General Solution : $y = Ae^{-4x} + Be^{2x}$

Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

2. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$

$y=e^{mx}$

$y=Ae^{3x}$ $y=Bxe^{3x}$

$m^2-6m+9=0$

$(m-3)(m-3)=0$

$m=3$

general Solution: $y=Ae^{3x}+Bxe^{3x}$

or $e^{3x}(A+Bx)$

Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

3. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$

$m^2+4m+13=0$

$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{16-52}}{2}=\frac{-4\pm\sqrt{-36}}{2}$

$m=\frac{-4\pm6i}{2}=\underset{p}{-2}\pm\underset{q}{3i}$

$y=e^{-2x}(A\cos3x+B\sin3x)$

Homogeneous 2nd order differential equations

Particular Solutions

Examples:

Find the particular solution of the second order differential equation

1. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=0$ given that when x = 0, y = 1 and $\frac{dy}{dx}=5$

$m^2-4m+3=0$

$(m-1)(m-3)=0$

$m=1,\ m=3$

$y=Ae^{x}+Be^{3x}$

$\boxed{y=2e^{3x}-e^{x}}$

When $x=0,\ y=1$

$1=Ae^0+Be^0$

$1=A+B$

$\frac{dy}{dx}=Ae^{x}+3Be^{x}$

$5=Ae^0+3Be^0$

$5=A+3B$

$1=A+B$

$4=2B$ $B=2$

$1=A+2$

$A=-1$

Homogeneous 2nd order differential equations

Particular Solutions

Examples:

Find the particular solution of the second order differential equation

2. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$ given that when x = 0, y = 1 and $\frac{dy}{dx}=2$

$m^2+4m+4=0$

$(m+2)(m+2)=0$

$m=-2$

$y=Ae^{2x}+Bxe^{-2x}$

$y=e^{-2x}+4xe^{-2x}$

or

$e^{-2x}(1+4x)$

$1=Ae^0+B(0)e^0$

$A=1$

$\frac{dy}{dx}=-2Ae^{-2x}+\left[Bx(-2e^{-2x})+e^{-2x}(B)\right]$

$=-2Ae^{-2x}-2Bxe^{-2x}+Be^{-2x}$

$2=-2(1)e^0-2B(0)e^0+Be^0$

$2=-2+B$

$B=4$

Homogeneous 2nd order differential equations

Particular Solutions

Examples:

Find the particular solution of the second order differential equation

3. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=0$ given that when x = 0, y = 2 and $\frac{dy}{dx}=1$

$m^2-4m+5=0$

$b^2-4ac<0 \Rightarrow$ no real roots

$m=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$

$m=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm2i}{2}=\underset{p}{2}\pm\underset{q}{1i}$

$y=e^{2x}(A\cos x+B\sin x)$

$\boxed{y=e^{2x}(2\cos x-3\sin x)}$

$y=e^{2x}A\cos x+e^{2x}B\sin x$

$2=e^0A\cos0+e^0B\sin0$

$2=A$

$\frac{dy}{dx}=e^{2x}(-A\sin x)+A\cos x(2e^{2x})$

$+Be^{2x}(\cos x)+\sin x(2Be^{2x})$

$1=-Ae^0\sin0+A\cos0(2e^0)$

$+Be^0(\cos0)+2Be^0\sin0$

$1=2A+B$

$1=4+B$

$B=-3$

Non-Homogeneous 2nd order differential equations

These are equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

where a, b and c are constants and f(x) ≠ 0.

$a\neq0$

- First treat it as if it were homogeneous and equate to zero to find the complementary function.
- Find a particular integral y_p (it will be in a similar form to f(x)).
- General Solution = Complementary Function + Particular Integral

$\boxed{y=y_c+y_p}$

Identifying the particular integral

- Linear, e.g. $f(x) = 2x + 1$ — $y_p = ax + b$
- Quadratic, e.g. $f(x) = x^2 - 1$ — $y_p = ax^2 + bx + c$
- e, e.g. $f(x) = 4e^{2x}$ — $y_p = Ae^{2x}$
- Trigonometric, e.g. $f(x) = 2\sin x + \cos x$ — $y_p = a\sin x + b\cos x$
- or e.g. $f(x) = 3\sin 2x$ — $y_p = a\sin 2x + b\cos 2x$

include because $3\sin 2x = 3\sin 2x + 0\cos 2x$

Non-Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

1. $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x + 1$$

Complementary Function: $m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2,\ m = -1$

$y_c: Ae^{2x} + Be^{-x}$

$\boxed{y = Ae^{2x} + Be^{-x} - 2x + \frac{1}{2}}$

Particular Integral

$y_p = ax + b$

$\frac{dy}{dx} = a \qquad \frac{d^2y}{dx^2} = 0$

$0 - a - 2(ax+b) = 4x + 1$

$-a - 2ax - 2b = 4x + 1$

equating coefficients: $-2a = 4$, $a = -2$

$-a - 2b = 1$

$2 - 2b = 1$

$-2b = -1$

$b = \frac{1}{2}$

$\boxed{y_p = -2x + \frac{1}{2}}$

Non-Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

2. $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 1$$

$y = y_p + y_c$

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

$y_c = Ae^{2x} + Bxe^{2x}$

$y = Ae^{2x} + Bxe^{2x} + \frac{1}{2}x^2 + x + 1$

Particular

$y_p: ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b \qquad \frac{d^2y}{dx^2} = 2a$

$2a - 4(2ax + b) + 4(ax^2 + bx + c) = 2x^2 + 1$

$2a - 8ax - 4b + 4ax^2 + 4bx + 4c = 2x^2 + 1 + 0x$

Equate coefficients

$4a = 2$, $a = \frac{1}{2}$

$-8a + 4b = 0$

$-4 + 4b = 0$

$b = 1$

$2a - 4b + 4c = 1$

$1 - 4 + 4c = 1$

$4c = 4$, $c = 1$

Non-Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

3. $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$$

$y_c: m^2 + 2m - 3 = 0$

$(m+3)(m-1) = 0$

$m = -3,\ m = 1$

$y_c: Ae^{-3x} + Be^{x}$

$y = Ae^{-3x} + Be^{x} + 2e^{2x}$

$y_p: pe^{2x}$

$\frac{dy}{dx} = 2pe^{2x} \qquad \frac{d^2y}{dx^2} = 4pe^{2x}$

$4pe^{2x} + 2(2pe^{2x}) - 3pe^{2x} = 10e^{2x}$

$5pe^{2x} = 10e^{2x}$

$p = 2$

Non-Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

4. $$\frac{d^2y}{dx^2} + 16y = e^{-2x}$$

$y_c: m^2 + 16 = 0$

$m^2 = -16$

$m = \pm 4i \qquad 0 \pm 4i$

$y_c: e^{0x}(A\sin 4x + B\cos 4x)$

$= A\sin 4x + B\cos 4x$

$y = A\sin 4x + B\cos 4x + \frac{1}{20}e^{-2x}$

$y_p: pe^{-2x}$

$\frac{dy}{dx} = -2pe^{-2x} \qquad \frac{d^2y}{dx^2} = 4pe^{-2x}$

$4pe^{-2x} + 16(pe^{-2x}) = e^{-2x}$

$20pe^{-2x} = e^{-2x}$

$20p = 1$

$p = \frac{1}{20}$

Non-Homogeneous 2nd order differential equations

Examples:

Find the general solution of the 2nd order differential equation

5. $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10\sin x$$

$y = y_c + y_p$

$y_c: m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$m = 3,\ m = 1$

$y_c: Ae^{3x} + Be^{x}$

$y = Ae^{3x} + Be^{x} + \sin x + 2\cos x$

$y_p: a\sin x + b\cos x$

$\frac{dy}{dx} = a\cos x - b\sin x$

$\frac{d^2y}{dx^2} = -a\sin x - b\cos x$

$-a\sin x - b\cos x - 4a\cos x + 4b\sin x + 3a\sin x + 3b\cos x = 10\sin x$

$= 2a\sin x + 2b\cos x - 4a\cos x + 4b\sin x = 10\sin x$

$2a + 4b = 10\ (\times 2) \qquad 4a + 8b = 20$

$-4a + 2b = 0 \qquad -4a + 2b = 0$

$2a + 8 = 10 \qquad 10b = 20$

$2a = 2 \qquad b = 2$

$a = 1$

Non-Homogeneous 2nd order differential equations

Particular Solution Example:

Find the particular solution of the 2nd order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

given that when x = 0, y = 2 and $\frac{dy}{dx} = 1$

$y_c:\ m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

$y_c:\ \underline{Ae^{2x} + Bxe^{2x}}$

$y_p:\ pe^x$

$\frac{dy}{dx} = pe^x \qquad \frac{d^2y}{dx^2} = pe^x$

$pe^x - 4pe^x + 4pe^x = e^x$

$pe^x = e^x$

$p = 1$

$y = \underline{Ae^{2x} + Bxe^{2x} + e^x}$

$\boxed{x=0, y=2}$

$2 = Ae^0 + B(0)e^0 + e^0$

$2 = A + 1$

$\underline{\underline{A = 1}}$

$\boxed{x=0,\ dy/dx = 1}$

$dy/dx = 2Ae^{2x} + Bx(2e^{2x}) + e^{2x}(B) + e^x$

$= 2Ae^{2x} + 2Bxe^{2x} + Be^{2x} + e^x$

$1 = 2Ae^0 + 2B(0)e^0 + Be^0 + e^0$

$1 = 2A + B + 1 \qquad 2A + B = 0$

$2 + B = 0$

$B = -2$

$y = e^{2x} - 2xe^{2x} + e^x$

$y = \underline{e^{2x}(1-2x) + e^x}$

Non-Homogeneous 2nd order differential equations

Ammending particular integrals

In some cases, the particular integral will already be part of the complementary function. You will then need to ammend it.

Examples:

1. Find the general solution of the 2nd order differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^{3x}$$

The x makes it different to the complementary function

$y_c:\ m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$

$m = 3,\ m = 2$

$y_c:\ \underline{Ae^{3x} + Be^{2x}}$

$y_p:\ pxe^{3x}$

$\frac{dy}{dx} = px(3e^{3x}) + e^{3x}(p)$

$= 3pxe^{3x} + pe^{3x}$

$\frac{d^2y}{dx^2} = 3px(3e^{3x}) + e^{3x}(3p) + 3pe^{3x}$

$= 9pxe^{3x} + 3pe^{3x} + 3pe^{3x}$

$= 9pxe^{3x} + 6pe^{3x}$

$9pxe^{3x} + 6pe^{3x} - 5(3pxe^{3x} + pe^{3x}) + 6pxe^{3x} = 2e^{3x}$

$\underline{9pxe^{3x}} + 6pe^{3x} - \underline{15pxe^{3x}} - 5pe^{3x} + \underline{6pxe^{3x}} = 2e^{3x}$

$pe^{3x} = 2e^{3x}$

$\underline{\underline{p = 2}}$

$y = Ae^{3x} \underline{+ Be^{2x} + 2xe^{3x}}$

Ammending Particular Integrals

Examples:

2. Find the general solution of the 2nd order differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 6e^{4x}$$

$y_c:\ m^2 - 8m + 16 = 0$

$(m-4)(m-4) = 0$

$\underline{m = 4}$

$y_c:\ \underline{Ae^{4x} + Bxe^{4x}}$

$y_p:\ px^2e^{4x}$

$\frac{dy}{dx} = px^2(4e^{4x}) + e^{4x}(2px)$

$= 4px^2e^{4x} + 2pxe^{4x}$

$\frac{d^2y}{dx^2} = 4px^2(4e^{4x}) + e^{4x}(8px) + 2px(4e^{4x}) + e^{4x}(2p)$

$= 16px^2e^{4x} + 8pxe^{4x} + 8pxe^{4x} + 2pe^{4x}$

$= 16px^2e^{4x} + 16pxe^{4x} + 2pe^{4x}$

$\underline{16px^2e^{4x}} + \underline{16pxe^{4x}} + 2pe^{4x} - \underline{32px^2e^{4x}} - \underline{16pxe^{4x}} + \underline{16px^2e^{4x}} = 6e^{4x}$

$2pe^{4x} = 6e^{4x}$

$\underline{\underline{p = 3}}$

$y = \underline{Ae^{4x} + Bxe^{4x} + 3x^2e^{4x}}$

Ammending Particular Integrals

Examples:

3. Find the general solution of the 2nd order differential equation

$$4\frac{d^2y}{dx^2} + 36y = \sin 3x$$

$y_c:\ 4m^2 + 36 = 0$

$4m^2 = -36$

$m^2 = -9$

$m = \pm 3i$

$0 \pm 3i$

$y_c:\ e^{0x}(A\sin 3x + B\cos 3x)$

$= \underline{A\sin 3x + B\cos 3x}$

sin 3x already in yc

$y_p:\ ax\sin 3x + bx\cos 3x$

$\frac{dy}{dx} = ax(3\cos 3x) + \sin 3x(a) + bx(-3\sin 3x) + \cos 3x(b)$

$= 3ax\cos 3x + a\sin 3x - 3bx\sin 3x + b\cos 3x$

$\frac{d^2y}{dx^2} = 3ax(-3\sin 3x) + \cos 3x(3a) + 3a\cos 3x - 3bx(3\cos 3x) + \sin 3x(-3b) - 3b\sin 3x$

$= -9ax\sin 3x + \underline{3a\cos 3x} + \underline{3a\cos 3x} - 9bx\cos 3x - \underline{3b\sin 3x} - \underline{3b\sin 3x}$

$= -9ax\sin 3x + 6a\cos 3x - 6b\sin 3x - 9bx\cos 3x$

$4(-9ax\sin 3x + 6a\cos 3x - 6b\sin 3x - 9bx\cos 3x) + 36(ax\sin x + bx\cos x) = \sin 3x$

$\cancel{-36ax\sin 3x} + 24a\cos 3x - 24b\sin 3x \cancel{- 36bx\cos 3x} + \cancel{36ax\sin x} + \cancel{36bx\cos x} = \sin 3x$

$-24b = 1 \qquad 24a = 0$

$b = -\frac{1}{24} \qquad a = 0$

$y = \underline{A\sin 3x + B\cos 3x - \frac{1}{24}x\cos 3x}$