(*a*) Obtain $\frac{dy}{dx}$ when y is defined as a function of x by the equation

$$y + e^y = x^2. \qquad 3$$

(*b*) Given $f(x) = \sin x \cos^3 x$, obtain $f'(x)$. 3

(a) $dy/dx + e^y \, dy/dx = 2x$

$dy/dx(1+e^y) = 2x$

$dy/dx = \frac{2x}{1+e^y}$

(b) $\sin x \cos^3 x$

$f'(x) = \sin x \left(3\cos^2 x(-\sin x)\right) + \cos^3 x \left(\cos x\right)$

$= -3\sin^2 x \cos^2 x + \cos^4 x$

Inverse Trigonometric Functions

For a function to have an inverse, a one-to-one correspondence has to exist.

E.g. If $f(x) = x^3$ then $f^{-1}(x) = \sqrt[3]{x}$ but if $f(x) = x^2$ then $f^{-1}(x) = \pm\sqrt{x}$ (Horizontal line test)

Sometimes an inverse function can exist if we restrict the domain. i.e. only allow certain values for x.

E.g. $f(x) = x^2$ can have an inverse so long as $x > 0$

Inverse Trigonometric Functions

Recall that when you work out x given sinx, there are endless solutions for x, therefore for the inverse to exist, we need to restrict the domain of the original function.

Try to think about how you could restrict the domain of sinx, cosx and tan x for $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ to exist

In many books, notes and software, $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ may be written as arcsin, arccos and arctan.

Differentiating Inverse Trigonometric Functions

To differentiate inverse trig. functions, we can use implicit differentiation.

1. $y = \sin^{-1}x$

$\sin y = x$

$x = \sin y$

$dx/dy = \cos y$

$dy/dx = \frac{1}{\cos y}$

$dy/dx = \frac{1}{\sqrt{1-x^2}}$

$dy/dx = \frac{1}{dx/dy}$

* We didn't use implicit differentiation on this one. Next example, we will instead!

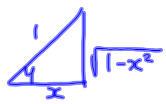
Differentiating Inverse Trigonometric Functions

2. $y = \cos^{-1}x$

$\cos y = x$

$-\sin y \, dy/dx = 1$

$dy/dx = -\frac{1}{\sin y}$



$dy/dx = -\frac{1}{\sqrt{1-x^2}}$

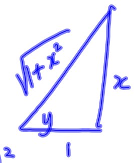
Differentiating Inverse Trigonometric Functions

3. y = tan⁻¹x

$\tan y = x$

$\sec^2 y \, \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = (\cos y)^2$



$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$

Differentiating Inverse Trigonometric Functions

Examples:

1. y = sin⁻¹(3x)

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \cdot 3$

$= \frac{3}{\sqrt{1-9x^2}}$

Differentiating Inverse Trigonometric Functions

Examples:

2. y = tan⁻¹(x²)

$\frac{dy}{dx} = \frac{1}{1+(x^2)^2} \cdot 2x$

$= \frac{2x}{1+x^4}$

Differentiating Inverse Trigonometric Functions

Examples:

3. y = cos⁻¹(1 – 2x)

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(1-2x)^2}} \cdot -2$

$= \frac{2}{\sqrt{1-(1-2x)^2}} = \frac{2}{\sqrt{1-(1-4x+4x^2)}}$

$= \frac{2}{\sqrt{4x-4x^2}} = \frac{2}{\sqrt{4x(1-x)}} = \frac{2}{2\sqrt{x(1-x)}} = \frac{1}{\sqrt{x(1-x)}}$

Differentiating Inverse Trigonometric Functions

Examples:

4. $y = \sin^{-1}\left(\frac{x}{5}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} \cdot \frac{1}{5}$

$= \frac{1}{5\sqrt{1-\frac{x^2}{25}}}$

$= \frac{1}{5\sqrt{\frac{25-x^2}{25}}} = \frac{1}{\frac{5\sqrt{25-x^2}}{\sqrt{25}}} = \frac{1}{\sqrt{25-x^2}}$

Differentiating Inverse Trigonometric Functions

Examples:

5. $y = \cos^{-1}(4x-1)$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(4x-1)^2}} \cdot 4$

$= -\frac{4}{\sqrt{1-(16x^2-8x+1)}}$

$= -\frac{4}{\sqrt{8x-16x^2}}$

$= -\frac{4}{\sqrt{4x(2-4x)}}$

$= -\frac{4}{2\sqrt{2x(1-2x)}}$

$= -\frac{2}{\sqrt{2x(1-2x)}}$

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$f(x) = \sqrt{x}\, e^{-x}$
$= x^{\frac{1}{2}} e^{-x}$

$f'(x) = x^{\frac{1}{2}}(-e^{-x}) + e^{-x}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$
$-e^{-x}x^{\frac{1}{2}} + \frac{1}{2}e^{-x}\frac{1}{\sqrt{x}}$
$= \frac{1}{e^x}\left(-\sqrt{x} + \frac{1}{2\sqrt{x}}\right)$
$= \frac{1}{e^x}\left(-\frac{2x+1}{2\sqrt{x}}\right)$
$= \frac{1}{2e^x\sqrt{x}}(-2x+1)$

$f(x) = (\cos^2 x) \cdot e^{\tan x}$

$f'(x) = \cos^2 x\left[\sec^2 x\, e^{\tan x}\right] + e^{\tan x}\left[2\cos x \sin x\right]$
$= e^{\tan x}\cos^2 x \sec^2 x - 2e^{\tan x}\cos x \sin x$
$= e^{\tan x}\left[\cos^2 x \frac{1}{\cos^2 x} - 2\cos x \sin x\right]$
$= e^{\tan x}\left[1 - \sin 2x\right]$

$\exp(\sin 2x) = e^{\sin 2x}$

Show that $1 + \tan^2 x = \sec^2 x$

L.H.S

$1 + \tan^2 x$

$1 + \frac{\sin^2 x}{\cos^2 x}$

$\frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x =$ R.H.S

$\boxed{\sec^2 x = \frac{1}{\cos^2 x}}$

$f'(x) = 3(x+1)(x-2)^2 + (x-2)^3$
$= (x-2)^2\left[3(x+1) + (x-2)\right]$
$= (x-2)^2\left[3x+3+x-2\right]$
$= (x-2)^2(4x+1) = 0$
$x = 2,\ x = -\frac{1}{4}$

Differentiating Inverse Trigonometric Functions

Using product and quotient rules

1. $f(x) = \ln x \tan^{-1}x$

$f'(x) = \ln x\left(\frac{1}{1+x^2}\right) + \tan^{-1}x \cdot \left[\frac{1}{x}\right]$
$= \frac{\ln x}{1+x^2} + \frac{\tan^{-1}x}{x}$

Differentiating Inverse Trigonometric Functions

Using product and quotient rules

2. $f(x) = x^2 \sin^{-1}x$

$f'(x) = x^2\left[\frac{1}{\sqrt{1-x^2}}\right] + \sin^{-1}x \cdot \left[2x\right]$
$= \frac{x^2}{\sqrt{1-x^2}} + 2x\sin^{-1}x$

Differentiating Inverse Trigonometric Functions

Using product and quotient rules

3. $f(x) = \sqrt{1-x^2}\cos^{-1}x$

$f(x) = (1-x^2)^{\frac{1}{2}}\cos^{-1}x$

$f'(x) = (1-x^2)^{\frac{1}{2}}\left[\frac{1}{-\sqrt{1-x^2}}\right] + \cos^{-1}x\left[\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot -2x\right]$

$= -\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

$-1 - \frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

2004 (b)

$g(x) = \frac{\tan^{-1}2x}{1+4x^2}$

$g'(x) = \frac{1+4x^2\left[\frac{1}{1+(2x)^2}\cdot 2\right] - \tan^{-1}2x \cdot [8x]}{(1+4x^2)^2}$

$= \frac{2 - 8x\tan^{-1}2x}{(1+4x^2)^2}$