Important things to note:

All the formulae from Higher should be known especially derivatives and integrals.

Keep a copy of AH formulae sheet close by for all topics.

We use a dot for multiplication.

Differentiating from first principles

Recall that the derivative of a function is the gradient of the tangent to the function at that point.

Given P(x, f(x)) and Q(x + h, f(x + h))

$m_{PQ} = \frac{f(x+h)-f(x)}{h}$ therefore as h tends to 0, $m_{PQ} = f'(x)$

We can differentiate any function by first principles.

Differentiating from first principles

Example:

$f(x) = 2x^2$

$f(x+h) = 2(x+h)^2$

$2(x^2+2xh+h^2) = 2x^2+4xh+2h^2$

$f(x+h)-f(x) \Rightarrow 2x^2+4xh+2h^2-2x^2$

$= 4xh+2h^2$

$\frac{f(x+h)-f(x)}{h} = \frac{4xh+2h^2}{h} = 4x+2h$

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 4x+2h = \underline{\underline{4x}}$

Differentiating from first principles

Differentiate $f(x) = 2x^2 - 3x + 1$ from first principles

Differentiate the following using the Chain Rule:

1. $(3x - 1)^3$

$= 3(3x-1)^2(3)$

$9(3x-1)^2$

2. 5cos3x

$-15\sin 3x$

3. $(2x^2 - 4)^{-5}$

$-5(2x^2-4)^{-6}(4x)$

$\underline{\underline{-20x(2x^2-4)^{-6}}}$

Differentiation – Recap. of the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

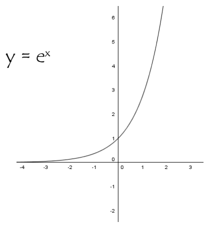
When differentiating using the chain rule, you are deriving the function and then multiplying by the derivative of the function within the function.

Examples: Differentiate

$y = (2 + x)^2$ $\qquad$ $y = 3\sin 2x$ $\qquad$ $f(x) = 2(2x^2 + 4)^3$

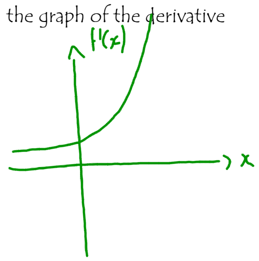
Differentiating e^x

$y = e^x$



Describe the gradient of this graph

Sketch the graph of the derivative



+m +m +m

Differentiating e^x

Given $f(x) = e^x$ then $f'(x) = e^x$

We differentiate exponential functions using the chain rule.

Examples: Differentiate the following

(a) e^{6x}

$f'(x) = e^{6x} \cdot 6$

$= \underline{6e^{6x}}$

(b) $f(x) = 3e^{x^2} + 4e^{-2x}$

$f'(x) = 3e^{x^2} \cdot (2x) + 4e^{-2x} \cdot -2$

$= \underline{6xe^{x^2} - 8e^{-2x}}$

(c) $y = e^{\sqrt{x^2+1}}$

$y = e^{(x^2+1)^{\frac{1}{2}}}$

$y = e^{(x^2+1)^{\frac{1}{2}}} \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$

$y = x(x^2+1)^{-\frac{1}{2}} e^{(x^2+1)^{\frac{1}{2}}}$

$= \dfrac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

(d) $y = 5x^2 + \dfrac{3}{e^{x^2}} = 5x^2 + 3e^{-x^2}$

$dy/dx = 10x + 3e^{-x^2} \cdot (-2x)$

$= 10x - 6xe^{-x^2}$

$= \underline{10x - \dfrac{6x}{e^{x^2}}}$

Differentiating $\log_e x$ or $\ln x$

Remember that $\ln x$ and e^x are inverse functions.

Therefore $e^{\ln x} = x$ and $\ln(e^x) = x$

$\boxed{f^{-1}(f(x)) = x}$

Let $y = \log_e x$

$x = e^y$

$dx/dy = e^y$

$dy/dx = \dfrac{1}{e^y} = \dfrac{1}{x}$

$\underline{d/dx \log_e x = \dfrac{1}{x}}$

Differentiating $\log_e x$ or $\ln x$

We differentiate the natural log using the chain rule also.

Examples: Differentiate the following

(a) $y = \ln 7x$

$dy/dx = \dfrac{1}{7x} \cdot 7$

$= \underline{\dfrac{1}{x}}$

(b) $y = \ln(3x - 1)$

$dy/dx = \dfrac{1}{3x-1} \cdot 3$

$= \dfrac{3}{3x-1}$

(c) $f(x) = \ln(5x^2 - 6)$

$f'(x) = \dfrac{1}{5x^2-6} \cdot 10x$

$= \underline{\dfrac{10x}{5x^2-6}}$

(d) $y = \ln(\cos 4x)$

$dy/dx = \dfrac{1}{\cos 4x} \cdot -4\sin 4x$

$= -\dfrac{4\sin 4x}{\cos 4x}$

$= \underline{-4\tan 4x}$

Differentiating $\log_e x$ or $\ln x$

(e) $y = \ln\left(\dfrac{2x+1}{x-1}\right)$

$y = \ln(2x+1) - \ln(x-1)$

$dy/dx = \dfrac{1}{2x+1} \cdot 2 - \dfrac{1}{x-1} \cdot 1$

$= \dfrac{2}{2x+1} - \dfrac{1}{x-1}$

$= \dfrac{2x-2-2x-1}{(2x+1)(x-1)} = \underline{\dfrac{-3}{(2x+1)(x-1)}}$

Differentiating $\log_e x$ or $\ln x$

(f) $y = \ln\left(\dfrac{\sqrt{x^2+1}}{(9x-4)^2}\right) = \ln(x^2+1)^{\frac{1}{2}} - \ln(9x-4)^2$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2+1}} \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) - \dfrac{1}{(9x-4)^2} \cdot 2(9x-4)(9)$

$= \dfrac{x}{x^2+1} - \dfrac{18(9x-4)}{(9x-4)^2}$

The Product Rule

Given two functions u(x) and v(x) where f(x) = u(x).v(x) then

$$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

First by the derivative of the second + second by the derivative of the first.

Proof:

Let $f(x) = u(x).v(x)$

$f(x+h) = u(x+h).v(x+h)$

$f(x+h) - f(x) = u(x+h).v(x+h) - u(x).v(x)$

$u(x+h).v(x+h) + u(x+h).v(x) - u(x+h).v(x) - u(x).v(x)$

$u(x+h)\left[v(x+h) - v(x)\right] + v(x)\left[u(x+h) - u(x)\right]$

$$\frac{f(x+h)-f(x)}{h} = \frac{u(x+h)\left[v(x+h)-v(x)\right]}{h} + \frac{v(x)\left[u(x+h)-u(x)\right]}{h}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{u(x+h)\left[v(x+h)-v(x)\right]}{h} + \lim_{h\to 0}\frac{v(x)\left[u(x+h)-u(x)\right]}{h}$$

$$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

The Product Rule

Examples: Differentiate the following

u v

1. y = (1 – x)³(3x + 1)²

$$\frac{dy}{dx} = (1-x)^3\,2(3x+1)(3) + (3x+1)^2\,3(1-x)^2(-1)$$

$$= 6(3x+1)(1-x)^3 - 3(1-x)^2(3x+1)^2$$

$$3(3x+1)(1-x)^2\left[2(1-x) - (3x+1)\right]$$

$$3(3x+1)(1-x)^2\left[2-2x-3x-1\right]$$

$$3(3x+1)(1-x)^2(1-5x)$$

$$3(1-5x)(3x+1)(1-x)^2$$

The Product Rule

2. y = x³sinx

$$\frac{dy}{dx} = x^3\cos x + \sin x\,(3x^2)$$

$$= x^3\cos x + 3x^2\sin x$$

$$x^2(x\cos x + 3\sin x)$$

The Product Rule

3. Given f(x) = cos2xcos4x, show that f′(x) = 2sin2x(1 – 6cos²2x)

$$f'(x) = \cos 2x(-4\sin 4x) + \cos 4x(-2\sin 2x)$$

$$= -4\cos 2x\sin 4x - 2\cos 4x\sin 2x$$

$$-4\cos 2x\left[2\sin 2x\cos 2x\right] - 2\cos 4x\sin 2x$$

$$-8\cos^2 2x\sin 2x - 2\cos 4x\sin 2x$$

$$2\sin 2x\left[-4\cos^2 2x - \cos 4x\right]$$

$$2\sin 2x\left[-4\cos^2 2x - \left[2\cos^2 2x - 1\right]\right]$$

$$2\sin 2x\left[-6\cos^2 2x + 1\right]$$

$$2\sin 2x\left[1 - 6\cos^2 2x\right]$$

as required.

The Product Rule

4. y = x⁴cos2x

$$\frac{dy}{dx} = x^4(-2\sin 2x) + \cos 2x\,(4x^3)$$

$$= -2x^4\sin 2x + 4x^3\cos 2x$$

$$= 2x^3\left(2\cos 2x - x\sin 2x\right)$$

The Product Rule

5. $y = x^2\sqrt{x^2+1}$

$$y = x^2(x^2+1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x^2\left(\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\right) + (x^2+1)^{\frac{1}{2}}(2x)$$

$$= x^3(x^2+1)^{-\frac{1}{2}} + 2x(x^2+1)^{\frac{1}{2}}$$

$$= \frac{x^3}{\sqrt{x^2+1}} + 2x\sqrt{x^2+1}$$

$$= \frac{x^3 + 2x(x^2+1)}{\sqrt{x^2+1}} = \frac{3x^3+2x}{\sqrt{x^2+1}} = \frac{x(3x^2+2)}{\sqrt{x^2+1}}$$

Starter 16.6.17

Find the derivative

1. $y = x\sin(x^2)$

$\frac{dy}{dx} = x\left[2x\cos(x^2)\right] + \sin(x^2)(1)$

$= 2x^2\cos(x^2) + \sin(x^2)$

2. $y = x^2\cos^4 x$

$x^2(\cos x)^4$

$\frac{dy}{dx} = x^2\left[4(\cos x)^3(-\sin x)\right] + \cos^4 x\,(2x)$

$= -4x^2\cos^3 x\sin x + 2x\cos^4 x$

$-2x\cos^3 x(2x\sin x - \cos x)$

The Product Rule

6. $y = e^x \sin 2x$

$\frac{dy}{dx} = e^x(2\cos 2x) + \sin 2x\,(e^x)$

$= e^x\left[2\cos 2x + \sin 2x\right]$

The Product Rule

7. $y = x^3 \ln x$

$\frac{dy}{dx} = x^3\left(\frac{1}{x}\right) + \ln x(3x^2)$

$= x^2 + 3x^2\ln x$

$= x^2(1 + 3\ln x)$

Using the Product Rule to differentiate Logs with other bases

Given $y = \log_{10} x$

$x = 10^y$

$\ln x = \ln 10^y$

$\ln x = y\ln 10$

$y = \frac{\ln x}{\ln 10} = \frac{1}{\ln 10}\cdot \ln x$

$\frac{dy}{dx} = \frac{1}{\ln 10}\cdot\left(\frac{1}{x}\right) + \ln x(0)$

$= \frac{1}{x\ln 10}$

1. $x^4(2x^3+1)^5 = y$

The Quotient Rule

2. Find $f'\left(\frac{\pi}{3}\right)$ given $f(x) = \cos x\sin^2 x$

$f'(x)$

The Quotient Rule

This rule allows us to differentiate functions of the form $f(x)=\dfrac{u(x)}{v(x)}$

$$f'(x)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Bottom by the derivative of the top - top by the derivative of the bottom all over the bottom squared.

Proof: If $f(x)=\dfrac{u(x)}{v(x)}$ then $f(x)=u(x)\left[v(x)\right]^{-1}$

Therefore $f'(x)=u(x)\,.\,-\left[v(x)\right]^{-2}.v'(x)+\left[v(x)\right]^{-1}.u'(x)$

$$=-\frac{u(x)v'(x)}{\left[v(x)\right]^2}+\frac{u'(x)}{v(x)}$$

$$=\frac{-u(x)v'(x)+u'(x)v(x)}{(v(x))^2}$$

$$=\frac{v(x)u'(x)-u(x)v'(x)}{\left[v(x)\right]^2}$$

The Quotient Rule

Examples:

1. $\dfrac{2x}{x^2+1}=y$

$$\frac{dy}{dx}=\frac{(x^2+1)(2)-2x(2x)}{(x^2+1)^2}=\frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$=\frac{2-2x^2}{(x^2+1)^2}=\frac{2(1-x^2)}{(x^2+1)^2}$$

The Quotient Rule

Examples:

2. $y=\dfrac{(x^2+1)^4}{x^3}$

$$\frac{dy}{dx}=\frac{x^3\left[4(x^2+1)^3(2x)\right]-(x^2+1)^4(3x^2)}{x^6}$$

$$=\frac{8x^4(x^2+1)^3-3x^2(x^2+1)^4}{x^6}$$

$$=\frac{x^2(x^2+1)^3\left[8x^2-3(x^2+1)\right]}{x^6}$$

$$=\frac{\cancel{x^2}(x^2+1)^3(5x^2-3)}{x^{\cancel{6}4}}$$

$$=\frac{(x^2+1)^3(5x^2-3)}{x^4}$$

The Quotient Rule

Examples:

3. $y=\dfrac{\sin x}{x^3}$

$$\frac{dy}{dx}=\frac{x^3(\cos x)-\sin x\,(3x^2)}{x^6}=\frac{x^3\cos x-3x^2\sin x}{x^6}$$

$$=\frac{x^2(x\cos x-3\sin x)}{x^6}=\frac{x\cos x-3\sin x}{x^4}$$

The Quotient Rule

Examples:

4. $y=\dfrac{\cos x}{\sin^2 x}$

$$\frac{dy}{dx}=\frac{\sin^2x(-\sin x)-\cos x(2\sin x\cos x)}{\sin^4x}$$

$$=\frac{-\sin^3x-2\sin x\cos^2x}{\sin^4x}=\frac{\sin x(-\sin^2x-2\cos^2x)}{\sin^4x}$$

$$=\frac{-\sin^2x-2\cos^2x}{\sin^3x}=\frac{-\sin^2x-2(1-\sin^2x)}{\sin^3x}=\frac{\sin^2x-2}{\sin^3x}$$

The Quotient Rule

Examples:

5. $y=\dfrac{x^2}{\sqrt{(x+4)^3}}=\dfrac{x^2}{(x+4)^{\frac{3}{2}}}$

$$\frac{dy}{dx}=\frac{(x+4)^{\frac{3}{2}}(2x)-x^2\left(\frac{3}{2}(x+4)^{\frac{1}{2}}\right)}{(x+4)^3}=\frac{2x(x+4)^{\frac{3}{2}}-\frac{3x^2}{2}(x+4)^{\frac{1}{2}}}{(x+4)^3}$$

$$=\frac{x(x+4)^{\frac{1}{2}}\left[2(x+4)-\frac{3x}{2}\right]}{(x+4)^3}=\frac{x(x+4)^{\frac{1}{2}}\left[\frac{1}{2}x+8\right]}{(x+4)^3}$$

$\times(x+4)^{-\frac{1}{2}}$

$\times(x+4)^{\frac{1}{2}}$

$$=\frac{x\left(\frac{1}{2}x+8\right)}{(x+4)^{5/2}}$$

$$=\frac{x(x+16)}{2(x+4)^{5/2}}$$

The Quotient Rule

Examples:

6. $$y = \frac{x^2}{e^{2x}}$$

$$dy/dx = \frac{e^{2x}(2x) - x^2(2e^{2x})}{(e^{2x})^2} = \frac{2xe^{2x} - 2x^2e^{2x}}{e^{4x}}$$

$$= \frac{2xe^{2x}(1-x)}{e^{4x}} = \underline{\underline{\frac{2x(1-x)}{e^{2x}}}}$$

The Quotient Rule

Examples:

7. $$y = \frac{\ln x}{e^{2x}}$$

$$dy/dx = \frac{e^{2x}\left(\frac{1}{x}\right) - \ln x(2e^{2x})}{e^{4x}} = \frac{\frac{e^{2x}}{x} - 2e^{2x}\ln x}{e^{4x}}$$

$$= \frac{e^{2x}\left(\frac{1}{x} - 2\ln x\right)}{e^{4x}} = \frac{\frac{1}{x} - 2\ln x}{e^{2x}} = \underline{\underline{\frac{1 - 2x\ln x}{xe^{2x}}}}$$