Starter 2.11.2017

(a) Differentiate $y = x\tan^{-1}2x$. **3**

(b) Given $f(x) = \frac{1-x^2}{1+4x^2}$, find $f'(x)$, simplifying your answer. **3**

(c) A curve is given by the parametric equations

$$x = 6t \text{ and } y = 1 - \cos t.$$

Find $\frac{dy}{dx}$ in terms of t. **2**

Recap. of Integration

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)(a)} + C \qquad n \neq -1$$

Recap. of Integration

Examples:

1. $\int \frac{3x^2-1}{\sqrt{x}}dx = \int \frac{3x^2-1}{x^{\frac{1}{2}}}dx = \int \frac{3x^2}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}}dx$

$$= \int \left(3x^{\frac{3}{2}} - x^{-\frac{1}{2}}\right)dx$$

$$= \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\underline{\underline{\frac{6x^{\frac{5}{2}}}{5} - 2x^{\frac{1}{2}} + C}}$$

Recap. of Integration

Examples:

2. $\int (2x+3)^5 dx$

$$= \frac{(2x+3)^6}{6(2)} + C$$

$$= \underline{\underline{\frac{(2x+3)^6}{12} + C}}$$

Recap. of Integration

Examples:

3. $\int_0^2 \sqrt{4x+1}dx = \int_0^2 (4x+1)^{\frac{1}{2}}dx$

$$= \left[\frac{(4x+1)^{\frac{3}{2}}}{\frac{3}{2}(4)}\right]_0^2 = \left[\frac{\left(\sqrt{4x+1}\right)^3}{6}\right]_0^2$$

$$= \frac{\left(\sqrt{4(2)+1}\right)^3}{6} - \frac{\sqrt{1}}{6}$$

$$= \frac{27}{6} - \frac{1}{6} = \frac{26}{6} = \frac{13}{3} = \underline{\underline{4\tfrac{1}{3}}}$$

Integrating exponential functions

$$\int e^x dx = e^x + C$$

$$\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$$

(This rule only applies to exponents with linear functions as powers)

Integrating exponential functions

Examples:

1. $\int 6e^{3x} + 4e^{-x} dx$

$= 6\left(\frac{1}{3}\right)e^{3x} + 4\left(\frac{1}{-1}\right)e^{-x} + C$

$2e^{3x} - 4e^{-x} + C$

Integrating exponential functions

Examples:

2. $\int \frac{8}{e^{2x}} dx = \int 8e^{-2x} dx$

$= 8\left(\frac{1}{-2}\right)e^{-2x} + C = -4e^{-2x} + C$

Integrating exponential functions

Examples:

3. $\int (e^x + \frac{1}{e^x})^2 dx = \int e^{2x} + 2 + \frac{1}{e^{2x}} \, dx$

$\int e^{2x} + 2 + e^{-2x} dx$

$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2e^{2x}} + C$

Integrating exponential functions

Examples:

4. Find the exact value of

$\int_0^1 \frac{(e^{2x}-1)^2}{e^x} dx = \int_0^1 \frac{e^{4x} - 2e^{2x} + 1}{e^x} dx$

$= \int_0^1 e^{3x} - 2e^x + e^{-x} dx$

$= \left[\frac{1}{3}e^{3x} - 2e^x - e^{-x}\right]_0^1 = \left[\frac{1}{3}e^3 - 2e - e^{-1}\right] - \left[\frac{1}{3}e^0 - 2e^0 - e^0\right]$

$= \frac{1}{3}e^3 - 2e - \frac{1}{e} - \frac{1}{3} + 2 + 1$

$= \frac{1}{3}e^3 - 2e - \frac{1}{e} + 2\frac{2}{3}$

Starter 3.11.2017

Given $f(x) = \frac{x-1}{1+x^2}$, show that $f'(x) = \frac{1+2x-x^2}{(1+x^2)^2}$. 3

Using logs in integration

Recall $\frac{d}{dx}\ln x = \frac{1}{x}$ x > 0 neccessary as lnx only defined when x > 0

$\int \frac{1}{x} dx = \ln|x| + C$

$\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

Using logs in integration

Examples:

1. $\int \frac{8}{2x+1}dx$

$= 8\left(\frac{1}{2}\right)\ln|2x+1| + C$

$4\ln|2x+1| + C$

Using logs in integration

Examples:

2. $\int\left(\frac{2}{x}+\frac{4}{6x+1}-\frac{6}{1-3x}\right)dx$

$= 2\ln|x| + 4\left(\frac{1}{6}\right)\ln|6x+1| - 6\left(-\frac{1}{3}\right)\ln|1-3x| + C$

$2\ln|x| + \frac{2}{3}\ln|6x+1| + 2\ln|1-3x| + C$

Using logs in integration

Examples:

3. Show that $\int_2^{14}\frac{1}{2x-1}dx = \ln 3$

$= \left[\frac{1}{2}\ln|2x-1|\right]_2^{14}$

$= \left[\frac{1}{2}\ln 27\right] - \left[\frac{1}{2}\ln 3\right] = \frac{1}{2}\left[\ln 27 - \ln 3\right]$

$\frac{1}{2}\ln\frac{27}{3} = \frac{1}{2}\ln 9 = \ln 9^{\frac{1}{2}} = \ln 3$ as required.

$\int_0^1\left(\frac{1}{2x+1} + \frac{1}{4x+1}\right)dx$

$= \left[\frac{1}{2}\ln|2x+1| + \frac{1}{4}\ln|4x+1|\right]_0^1$

$= \left[\frac{1}{2}\ln 3 + \frac{1}{4}\ln 5\right] - \left[\frac{1}{2}\ln 1 + \frac{1}{4}\ln 1\right]$

$= \frac{1}{2}\ln 3 + \frac{1}{4}\ln 5 - \frac{3}{4}\ln 1$

$= \frac{1}{2}\ln 3 + \frac{1}{4}\ln 5 - \ln 1$

$= \frac{1}{2}\ln 3 + \frac{1}{4}\ln 5$

$\frac{2\ln 3 + \ln 5}{4} = \frac{\ln 9 + \ln 5}{4} = \frac{\ln 45}{4}$

Integrating trigonometric functions

Recall: $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

we now also can say

$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$

Note: Integrating $\sin^2 x$ and $\cos^2 x$ requires trigonometric identities:

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$ $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Integrating trigonometric functions

Examples:

1. $\int(3\cos x - 4\sin 2x)dx$

$3\sin x - 4\left(\frac{1}{-2}\right)\cos 2x + C$

$3\sin x + 2\cos 2x + C$

Integrating trigonometric functions

Examples:

2. $\int 6\sec^2 4x dx$

$= \frac{6}{4}\tan 4x + C$

$= \frac{3}{2}\tan 4x + C$

Integrating trigonometric functions

Examples:

3. Show that

$$\int_0^{\frac{\pi}{6}} (\sin 2x + \sec^2 x)dx = \frac{4\sqrt{3}+3}{12}$$

$= \left[-\frac{1}{2}\cos 2x + \tan x\right]_0^{\frac{\pi}{6}}$

$= \left[-\frac{1}{2}\cos\frac{\pi}{3} + \tan\frac{\pi}{6}\right] - \left[-\frac{1}{2}\cos 0 + \tan 0\right]$

$= \left[-\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{3}}\right] - \left[-\frac{1}{2}\right]$

$= -\frac{1}{4} + \frac{1}{\sqrt{3}} + \frac{1}{2} = \frac{1}{4} + \frac{\sqrt{3}}{3} = \frac{3}{12} + \frac{4\sqrt{3}}{12} = \frac{3+4\sqrt{3}}{12}$ as required

Integrating trigonometric functions

Some functions require simplifying first.

Examples:

4. Evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^3 x \operatorname{cosec}^2 x dx$$