Integration by substitution

We use substitution to integrate when we have more than one fucntion e.g. products or quotients that are difficult to simplify. The function to be replaced must be similar to the other function when differentiated.

(You will be given the substitution)

Examples: Integrate the following

1. $\int x(x^2+1)^5\,dx$ Let $u = x^2 + 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int \frac{1}{2}u^5\,du$

$\left[\frac{1}{2}\frac{u^6}{6} + C\right]$

$= \frac{u^6}{12} + C$

$= \frac{(x^2+1)^6}{12} + C$

Integration by substitution

2. $\int x^2\sqrt{1-x^3}\,dx$ let $u = 1 - x^3$

$\frac{du}{dx} = -3x^2$

$du = -3x^2\,dx$

$-\frac{1}{3}du = x^2\,dx$

$\int -\frac{1}{3}u^{\frac{1}{2}}\,du$

$= -\frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= -\frac{1}{3}\frac{2u^{3/2}}{3} + C$

$= -\frac{2u^{3/2}}{9} + C \rightarrow \frac{-2\sqrt{1-x^3}^{\,3}}{9} + C$

Integration by substitution

3. $\int \sin^3 x\cos x\,dx$ let $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$\int u^3\,du$

$= \frac{u^4}{4} + C$

$= \frac{\sin^4 x}{4} + C$

Integration by substitution

4. $\int \frac{x}{3x^2+1}dx$ let $u = 3x^2 + 1$

$\int \frac{x}{1} \cdot \frac{1}{3x^2+1}\,dx$

$\frac{du}{dx} = 6x$

$du = 6x\,dx$

$\frac{1}{6}du = x\,dx$

$= \int \frac{1}{6}\frac{1}{u}\,du$

$= \frac{1}{6}\ln|u| + C$

$= \frac{1}{6}\ln|3x^2+1| + C$

Integration by substitution

5. $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}dx$ let $u = 1 + \cos 2x$

$\frac{du}{dx} = -2\sin 2x$

$du = -2\sin 2x\,dx$

$-\frac{1}{2}du = \sin 2x\,dx$

$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= -\frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = -u^{\frac{1}{2}} + C = -\sqrt{1+\cos 2x} + C$

Integration by substitution

6. $\int \frac{x^2}{(x^3+1)^5}dx$ let $u = x^3 + 1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$= \frac{1}{3}\int \frac{1}{u^5}\,du$

$= \frac{1}{3}\int u^{-5}\,du = \frac{1}{3}\left[\frac{u^{-4}}{-4}\right] + C = -\frac{1}{12u^4} + C$

$= -\frac{1}{12(x^3+1)^4} + C$

Integration by substitution

7. $\int \cos x e^{\sin x}\,dx$

let u = sinx

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$\int e^u\,du$

$= e^u + c$

$= e^{\sin x} + c$

Integration by substitution

8. $\int x(x+1)^3\,dx$

let u = x + 1

$\frac{du}{dx} = 1$

$du = 1\,dx$

$u = x+1$

$u - 1 = x$

$\int (u-1)u^3\,du$

$\int u^4 - u^3\,du$

$= \frac{u^5}{5} - \frac{u^4}{4} + C$

$\frac{(x+1)^5}{5} - \frac{(x+1)^4}{4} + C$

Integration by substitution

9. $\int \frac{x+1}{\sqrt{x-2}}\,dx$

let u = x − 2

$\frac{du}{dx} = 1$

$\boxed{du = 1\,dx}$

$u = x - 2$

$\boxed{x = u + 2}$

$\int \frac{u+2+1}{u^{\frac{1}{2}}}\,du$

$= \int \frac{u+3}{u^{\frac{1}{2}}}\,du = \int \frac{u}{u^{\frac{1}{2}}} + \frac{3}{u^{\frac{1}{2}}}\,du = \int u^{\frac{1}{2}} + 3u^{-\frac{1}{2}}\,du$

$\frac{2u^{\frac{3}{2}}}{3} + \frac{3u^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{2}{3}u^{\frac{3}{2}} + 6u^{\frac{1}{2}} + C$

$= \frac{2}{3}(x-2)^{\frac{3}{2}} + 6(x-2)^{\frac{1}{2}} + C$

Integration by substitution

10. $\int x^3\sqrt{x^2+1}\,dx$

let u = x² + 1

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\boxed{\frac{1}{2}du = x\,dx}$

$u = x^2 + 1$

$\boxed{x^2 = u - 1}$

$\frac{1}{2}\int (u-1)(u)^{\frac{1}{2}}\,du$

$= \frac{1}{2}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

$= \frac{1}{2}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{1}{2}\left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3}\right] + C$

$= \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3} + C = \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{(x^2+1)^{\frac{3}{2}}}{3} + C$

Evaluating definite integrals – Integration by substitution

When evaluating definite integrals using substitution, we don't have to change back the original 'x' provided we change the limits at the point of substitution.

Examples: Evaluate the following

1. $\int_0^1 x^3(x^4+1)^3\,dx$

Let u = x⁴ + 1

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

Changing limits

$u = x^4 + 1$

$u = 1^4 + 1 = 2$

$u = 0^4 + 1 = 1$

$= \frac{1}{4}\int_1^2 u^3\,du$

$= \frac{1}{4}\left[\frac{u^4}{4}\right]_1^2$

$= \frac{1}{4}\left[\frac{2^4}{4}\right] - \frac{1}{4}\left[\frac{1^4}{4}\right] = 1 - \frac{1}{16} = \frac{15}{16}$

Evaluating definite integrals – Integration by substitution

2. $\int_0^{\frac{\pi}{3}} \tan x\,dx$

Let u = cosx

$\frac{du}{dx} = -\sin x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

Change limits

$u = \cos x = \cos\frac{\pi}{3} = \frac{1}{2}$

$u = \cos 0 = 1$

$\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx = \int_1^{\frac{1}{2}} -\frac{1}{u}\,du = \left[-\ln|u|\right]_1^{\frac{1}{2}}$

$= -\ln\frac{1}{2} + \ln 1$

$= -\ln\frac{1}{2}$

$= -\ln 2^{-1} = \ln 2$

Evaulating definite integrals – Integration by substitution

3. $\int_2^5 x\sqrt{x-1}\,dx$

Let u = x – 1

$\frac{du}{dx} = 1$

$\boxed{du = 1dx}$

$u = x - 1$

$\boxed{x = u+1}$

Change limits

$u = 5-1 = 4$

$u = 2-1 = 1$

$\int_1^4 (u+1)u^{\frac{1}{2}}\,du$

$\int_1^4 u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du$

$$= \left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right]_1^4 = \left[\frac{2\sqrt{4}^5}{5} + \frac{2\sqrt{4}^3}{3}\right] - \left[\frac{2}{5} + \frac{2}{3}\right]$$

$$= \frac{64}{5} + \frac{16}{3} - \frac{2}{5} - \frac{2}{3} = \frac{62}{5} + \frac{14}{3} = \frac{186}{15} + \frac{70}{15} = \frac{256}{15} = 17\frac{1}{15}$$

Evaulating definite integrals – Integration by substitution

4. $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

Let u = sinx

$\frac{du}{dx} = \cos x$

$\boxed{du = \cos x\,dx}$

$\cos^3 x = 1 - \sin^2 x$

$\boxed{\cos^2 x = 1 - u^2}$

Limits new

$u = \sin\frac{\pi}{2} = 1$

$u = \sin 0 = 0$

$$= \int_0^1 1 - u^2\,du = \left[u - \frac{u^3}{3}\right]_0^1 = \left[1 - \frac{1}{3}\right] - [0] = \frac{2}{3}$$

5. $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$

Let u = 1 + x^2

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\boxed{\frac{1}{2}du = x\,dx}$

$u = 1 + x^2$

$\boxed{x^2 = u - 1}$

New limits

$u = 1 + 1^3 = 2$

$u = 1 + 0 = 1$

$\frac{1}{2}\int_1^2 \frac{u-1}{u^4}\,du$

$$= \frac{1}{2}\int_1^2 \frac{u}{u^4} - \frac{1}{u^4}\,du = \frac{1}{2}\int_1^2 u^{-3} - u^{-4}\,du$$

$$= \frac{1}{2}\left[\frac{u^{-2}}{-2} + \frac{u^{-3}}{3}\right]_1^2 = \frac{1}{2}\left[-\frac{1}{2(2)^2} + \frac{1}{3(2)^3}\right] - \frac{1}{2}\left[-\frac{1}{2} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{8} + \frac{1}{24}\right] - \frac{1}{2}\left[-\frac{1}{6}\right] = -\frac{1}{24} + \frac{1}{12} = \frac{1}{24}$$

Evaulating definite integrals – Integration by substitution

5. $\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx$ Let u = 1 + x^2

Integration by substitution – Using Trig. Identities

Need to remember:

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\sin 2x = 2\sin x\cos x$

$2\cos^2 x - 1 = \cos 2x$

$2\cos^2 u = \cos 2u + 1$

Integration by substitution – Using Trig. Identities

Examples:

1. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Let $x = \sin u$

$\frac{dx}{du} = \cos u$

$dx = \cos u\,du$

Change limits

$1 = \sin u$

$u = \sin^{-1}(1) = \frac{\pi}{2}$

$0 = \sin u$

$u = \sin^{-1}(0) = 0$

$$\int_0^{\frac{\pi}{2}} \frac{\cos u}{\sqrt{1-\sin^2 u}}\,du = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\cos u}\,du$$

$$= \int_0^{\frac{\pi}{2}} 1\,du = [u]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Integration by substitution – Using Trig. Identities

Examples:

2. $\int \sqrt{4-x^2}\,dx$

Let $x = 2\sin u$

$\frac{dx}{du} = 2\cos u$

$\boxed{dx = 2\cos u\,du}$

$$\int \sqrt{4-(2\sin u)^2}\cdot 2\cos u\,du \rightarrow \int \sqrt{4 - 4\sin^2 u}\cdot 2\cos u\,du$$

$$\int \sqrt{4(1-\sin^2 u)}\cdot 2\cos u\,du = \int \sqrt{4}\sqrt{1-\sin^2 u}\cdot 2\cos u\,du$$

$$\int 2\sqrt{\cos^2 u}\cdot 2\cos u\,du = \int 4\cos^2 u\,du$$

$\cos 2x = 2\cos^2 x - 1$

$\boxed{\cos^2 x = \frac{1}{2}(\cos 2x + 1)}$

$$= \int 4\left(\frac{1}{2}(\cos 2u + 1)\right)du$$

$$= \int 2\cos 2u + 2\,du$$

$$= 2\left(\frac{1}{2}\right)\sin 2u + 2u + C$$

$$= \sin 2u + 2u + C$$

$$= 2\sin u\cos u + 2u + C$$

$$= x\sqrt{1 - \frac{x^2}{4}} + 2\sin^{-1}\left(\frac{x}{2}\right) + C$$

$\cos^2 u + \sin^2 u = 1$

$\cos^2 u = 1 - \sin^2 u$

$\cos u = \sqrt{1 - \sin^2 u}$

$\cos u = \sqrt{1 - \frac{x^2}{4}}$

$\boxed{x = 2\sin u}$

$\sin u = \frac{x}{2}$

$u = \sin^{-1}\left(\frac{x}{2}\right)$

2002 A$(-1,5)$ HW

$x = t^2+t-1 \quad , \quad y = 2t^2 - t + 2$

$-1 = t^2+t-1$
$t^2+t=0$
$t(t+1)=0$
$t=0 \quad t=-1$

$y = 2(0)^2 - 0 + 2 = 2$
$y = 2(-1)^2 - (-1) + 2 = \underline{\underline{5}}$

Therefore $(-1,5)$ lies on curve

Show $1+\cot^2\theta = \mathrm{cosec}^2\theta$ HW

L.H.S
$= 1 + \frac{1}{\tan^2\theta}$
$= 1 + \frac{\cos^2\theta}{\sin^2\theta}$
$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}$
$= \underline{\underline{\frac{1}{\sin^2\theta} = \mathrm{cosec}^2\theta}}$

$\boxed{x = \cot y}$ dy/dx

$1 = -\mathrm{cosec}^2 y \, dy/dx$
$-\frac{1}{\mathrm{cosec}^2 y} = \frac{dy}{dx}$
$\frac{-1}{1+\cot^2 y} = \frac{dy}{dx}$
$\underline{\underline{\frac{-1}{1+x^2} = dy/dx}}$

HW

$dy/dx = 6t^{2\frac{1}{2}} - 10t^{1\frac{1}{2}}$

$d^2y/dx^2 = \frac{d}{dt}\left(dy/dx\right) \cdot dt/dx$

$\left(15t^{1\frac{1}{2}} - 15t^{\frac{1}{2}}\right) 2t^{\frac{1}{2}}$

$\underline{30t^2 - 30t}$

$dx/dt = \frac{1}{2}t^{-\frac{1}{2}}$
$\frac{dx}{dt} = \frac{1}{2t^{\frac{1}{2}}}$
$\frac{dt}{dx} = 2t^{\frac{1}{2}}$

$d^2y/dx^2 = 0$ at pt. Inflection
$30t^2 - 30t = 0$
$30t(t-1) = 0$
$\cancel{t=0} \quad t=1$

$(1, -\frac{3}{2})$
$m = -4$

$dy/dx = m = 6t^{2\frac{1}{2}} - 10t^{\frac{1}{2}}$
at $t=1 \quad m = 6 - 10 = -4$
$y = t^3 - \frac{5}{2}t^2 = 1 - \frac{5}{2} = -\frac{3}{2}$
$x = \sqrt{t} = \sqrt{1} = 1$

Integration by substitution – Using Trig. Identities

Examples:

3. $\displaystyle\int_0^{\frac{5}{\sqrt{2}}} \sqrt{25-x^2}\,dx$

Integration by substitution – Using Trig. Identities

Examples:

2014 Q12.

Use the substitution $x = \tan\theta$ to determine the exact value of

$$\int_0^1 \frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}.$$

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$x = \tan\theta$
$\frac{dx}{d\theta} = \sec^2\theta$
$dx = \sec^2\theta\, d\theta$

Change limits
$1 = \tan\theta$
$\theta = \tan^{-1} 1 = \frac{\pi}{4}$
$0 = \tan\theta$
$\theta = \tan^{-1} 0 = 0$

$\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{(\sec\theta)^2}{(\sec\theta)^3}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta}\, d\theta$

$\left(\sqrt{\sec^2\theta}\right)^3 = \sec^3\theta$

$= \displaystyle\int_0^{\frac{\pi}{4}} \cos\theta\, d\theta = \left[\sin\theta\right]_0^{\frac{\pi}{4}}$

$= \sin\frac{\pi}{4} - \sin 0 = \underline{\underline{\frac{1}{\sqrt{2}}}}$

Integration by substitution – A special case

Sometimes you will come across a fraction where the top is the derivative of the bottom.

E.g.

$\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\, dx$

$u = \sin x$
$\frac{du}{dx} = \cos x$
$du = \cos x\, dx$

$= \displaystyle\int \frac{1}{u}\, du$
$= \ln|u| + C$
$= \underline{\underline{\ln|\sin x| + C}}$

Integration by substitution – A special case

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Examples:

1. $\int \frac{2e^{2x}}{1+e^{2x}}dx$

$= \ln|1+e^{2x}| + c$

2. $\int \frac{18x^2 - 4x + 10}{6x^3 - 2x^2 + 10x + 7}dx$

$= \ln|6x^3 - 2x^2 + 10x + 7| + C$

Integration by substitution – A special case

Sometimes we may need to manipulate or take out a common factor first.

3. $\int \frac{12x+8}{3x^2+4x-8}dx = 2\int \frac{6x+4}{3x^2+4x-8}dx$

$= 2\ln|3x^2+4x-8| + C$