3rd Unit Assessment Topics:

- Functions - graphs, asymptotes, modulus etc.
- S.P.'s, points of inflection, nature
- Differentiation and Integration in context (Velocity, acceleration, area and volume etc.)
- Proofs

1.5 Applying algebraic and geometric skills to methods of proof	
Sub-skill	**Description**
Disproving a conjecture by providing a counter-example	Disprove a conjecture by providing a counter-example Know and be able to use the symbols ∃ and ∀ Write down the negation of a statement
Using indirect or direct proof in straightforward examples	Prove a statement by contradiction Use further proof by contradiction Use proof by contraposition Use direct proof in straightforward examples

Definitions

- A conjecture is some mathematical statement, true or false
  eg $4 + 3 = 7$ (true) $2 + 8 > 12$ (false)
- The negation of a statement is the opposite statement
  eg the negation of the two statements above are:
  $4 + 3 \neq 7$ (false) $2 + 8 < 12$ (true)
- A proof is a way of showing that a mathematical statement is true
- A theorem(theory) is a mathematical conjecture which has been proved

Notation

$x \Rightarrow y$ means x implies y

$x \Leftarrow y$ means that x is implied by y

$\Leftrightarrow$ or 'iff' means if and only if

$\forall$ x means for all x

$\exists$ means there exists an x

$\therefore$ means therefore

Direct Proof

This is all about proving that the Left Hand Side equals the Right Hand Side.

- Always work algebraically for these.
- Let even numbers be 2k
- Let odd numbers be 2k + 1

Examples:

1. Prove that the product of 2 even numbers is always even.

Let the two numbers be $2a$ and $2b$

Product $= 2a \times 2b = 4ab$
$= 2(2ab)$ which is divisible by 2 for all a, b therefore even.

Direct Proof

Examples:

2. Prove that if $x > 1$, then $x^2 > 1$

let $x = m+1$ where $m \geq 1$

$x^2 = (m+1)^2 = m^2 + 2m + 1 \geq 1$ because $m \geq 1$

therefore if $x > 1$, then $x^2 > 1$

Direct Proof

Examples:

3. Prove that a number under 100 is divisible by 3 if the sum of its digits is divisible by 3.

Let q be the number $\quad q = 10a + b \quad a, b \leq 9$

but $\underline{a + b \text{ is divisible by } 3}$ let sum of its digits $= 3m$

$a + b = 3m$

$q = 10a + b = 9a + a + b$

$= 9a + 3m$

$= \underline{\underline{3(3a+m)}}$ which is divisible by 3

Direct Proof

Examples:

4. Prove that $a \neq b$ if $a^2 + b^2 > 2ab \qquad a, b \in Z$

$a^2 + b^2 > 2ab$

$a^2 + b^2 - 2ab > 0$

$a^2 - 2ab + b^2 > 0$

$(a-b)^2 > 0$

$a - b > 0 \quad a - b < 0$

$a > b \quad a < b$

$a \neq b$ as required

Direct Proof

Examples:

5. Prove that the square of an odd integer is odd

let integer $= 2m+1$ where $m \in Z$

$(2m+1)^2 = 4m^2 + 4m + 1$

$= \underbrace{2(2m^2+2m)}_{\text{even}} + 1 \Rightarrow \underline{\underline{2(2m^2+2m)+1}}$ is odd

Proof by Contradiction

Proof by contradiction assumes the conjecture to be false and shows that this leads to the deduction of a false conclusion and so the original conjecture must therefore have been true.

Examples:

1. Prove by contradiction that $\sqrt{2}$ is not a rational number

Assume $\sqrt{2}$ is rational

$\therefore \sqrt{2} = \frac{a}{b} \qquad a, b \in Z^+ \quad a, b$ have no common factors

$2 = \frac{a^2}{b^2}$

$2b^2 = a^2 \quad \Rightarrow a^2$ is even $\Rightarrow a$ is even

let $a = 2m \quad m \in Z$

$2b^2 = (2m)^2$

$2b^2 = 4m^2$

$b^2 = 2m^2 \quad \Rightarrow b^2$ is even $\therefore b$ is even

$\Rightarrow a$ and b both even

but $a, b \in Z^+$ have no common factors

$\Rightarrow$ contradiction

$\therefore \sqrt{2}$ is not rational

Proof by Contradiction

2. Prove by contradiction that if m^2 is even, then m is even

Assume that the above conjecture is false, so if m^2 is even then m is odd.

Let $m = 2a + 1$ where $a \in Z$

$m^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$ which is odd. This contradicts the assumption that m^2 is even so the conjecture is true.

Proof by Contradiction

3. Prove by contradiction that for any integer n, if n^2 is a multiple of 3, then n itself is a multiple of 3.

Assume untrue

$n^2 = 3m$ then $n = 3m+1 \qquad m \in Z$

or $n = 3m+2$

1. $n = 3m+1$

$n^2 = (3m+1)^2 = 9m^2 + 6m + 1 = 3(3m^2+2m) + 1$ which is not a multiple of 3

but n^2 is a multiple of 3 and this contradicts the assumption.

2. $n = 3m+2$

$n^2 = (3m+2)^2 = 9m^2 + 12m + 4 = 9m^2 + 12m + 3 + 1 = 3(3m^2 + 4m + 1) + 1$

which is not a multiple of 3 which contradicts the assumption.

=> The conjecture is true and for any integer n, if n^2 is a multiple of 3, then n is a multiple of 3.

Proof by counter-example

This is a way of showing that a statement cannot be correct by giving an example that shows the statement is not correct.

Examples:

1. $5x > 4x \quad \forall x \in W$

let $x = 0$

$\Rightarrow 5x = 5(0) = 0$

$4x = 4(0) = 0$

$0 \not> 0$ ∴ Statement is false

$5x = 4x$

Proof by counter-example

Examples:

2. $2x \neq 2^x \quad \forall x \in R$

let $x = 2$

$\Rightarrow 2x = 2(2) = 4$

$2^x = 2^2 = 4$

$\Rightarrow 2x = 2^x \Rightarrow$ Statement is false

Proof by counter-example

Examples:

3. $ax = bx \Rightarrow a = b \quad \forall x \in R$

let $x = 0$ $a = 1$, $b = 2$

$\Rightarrow ax = 1 \times 0$

$bx = 2 \times 0$

$1 \times 0 = 2 \times 0$

but $1 \neq 2$ therefore $a \neq b$ and statement is false

Proof by counter-example

Examples:

4. $\sqrt{1 - \sin^2 x} = \cos x \quad \forall x \in R$

Proof using contrapositive

Recall Pythagoras' Theorem:

If a triangle is right-angled at C then

$$c^2 = a^2 + b^2$$

This is known as an implication as it has an 'if A, then B' statement.

The contrapositive of this statement is "If $c^2 \neq a^2 + b^2$ then the triangle is not right-angled at C"

An implication and its contrapositive are logically equivalent. i.e. If the implication is true, then the contrapositive is true and vice versa.

This means that if you can prove that the contrapositive of a statement is true, then the original statement must also be true.

Proof using contrapositive

Examples:

1. Prove that for $x \in Z$, if $5x + 9$ is even, then x is odd.

Contrapositive: If x is $\underbrace{\text{not odd}}_{\text{even}}$, then $5x+9$ is $\underbrace{\text{not even}}_{\text{odd}}$

let $x = 2m$

then $5x+9 = 5(2m)+9 = 10m+9 = 2(5m+4)+1$

which is odd

Therefore if x is even, then 5x + 9 is odd.

As the contrapositive has been proven true, then the original statement must also be true.

Proof using contrapositive

Examples:

2. Prove that for n ∈ Z, if n^2 is odd, then n is odd.

Contrapositive : If n is ~~not odd~~ even then n^2 is ~~not odd~~ even

let $n = 2m$
then $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$ which is even

Therefore when n is even, n^2 is even. As the contrapositive has been proven true, then the original statement must also be true.

Proof using contrapositive

Past Paper Question 2017

13. Let n be an integer.

Using proof by contrapositive, show that if n^2 is even, then n is even. **4**

Proof using contrapositive

Past Paper Question Specimen Paper

12. (a) Given that m and n are positive integers state the negation of the statement:

m is even or n is even. **1**

(b) By considering the contrapositive of the following statement:

if mn is even then m is even or n is even,

prove that the statement is true for all positive integers m and n. **3**

Past Paper Questions

12. Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8. **3**

Past Paper Questions

8. (*a*) Prove that the product of two odd integers is odd. **2**

Past Paper Questions

11. For each of the following statements, decide whether it is true or false and prove your conclusion.

A For all natural numbers m, if m^2 is divisible by 4 then m is divisible by 4.

B The cube of any odd integer p plus the square of any even integer q is always odd. **5**