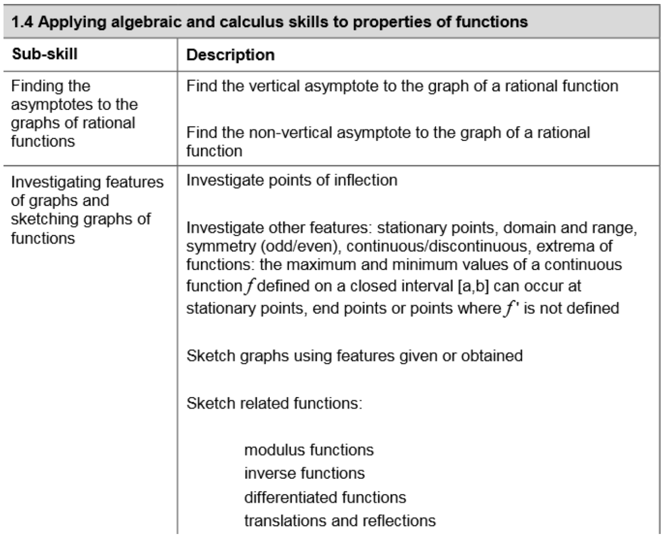


1.4 Applying algebraic and calculus skills to properties of functions	
Sub-skill	**Description**
Finding the asymptotes to the graphs of rational functions	Find the vertical asymptote to the graph of a rational function Find the non-vertical asymptote to the graph of a rational function
Investigating features of graphs and sketching graphs of functions	Investigate points of inflection Investigate other features: stationary points, domain and range, symmetry (odd/even), continuous/discontinuous, extrema of functions: the maximum and minimum values of a continuous function f defined on a closed interval [a,b] can occur at stationary points, end points or points where f' is not defined Sketch graphs using features given or obtained Sketch related functions: modulus functions inverse functions differentiated functions translations and reflections

Functions

Rational functions are of the form $\frac{p(x)}{q(x)}$ where p(x) and q(x) are polynomials.

If you can sketch a function without lifting the pencil off the paper, it is continuous.

E.g. $y = x^2$

If there is a break in the curve at any point, then it is said to be discontinuous.

E.g. y = tanx

Vertical Asymptotes of Functions

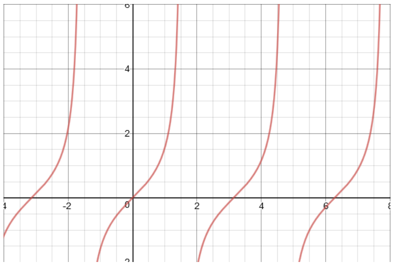
Not all values of 'x' will work in a function. When a value doesn't work we say the function is undefined for that particular value of 'x'.

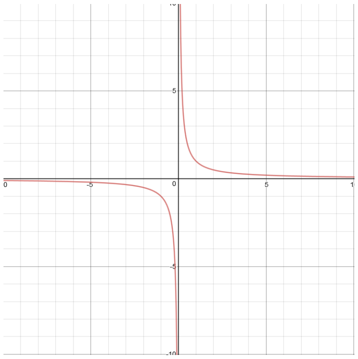
For f(x) = tan x, the function is undefined when x = 90°

For $f(x) = \frac{1}{x}$, the function is undefined when x = 0

x = 90 and x = 0 are equations of vertical lines .

We call these lines <u>vertical asymptotes</u>. The graph will get progressively closer to these lines but will **not** meet them. This causes discontinuities with the function.





Vertical Asymptotes of Functions

To find the vertical asymptotes, work out the values of x which make the function undefined. i.e. when the denominator = 0.

Examples:

1. $y = \frac{3}{2x-8}$

Vertical asymptote ⇒ $2x-6=0$

$\underline{x=4}$

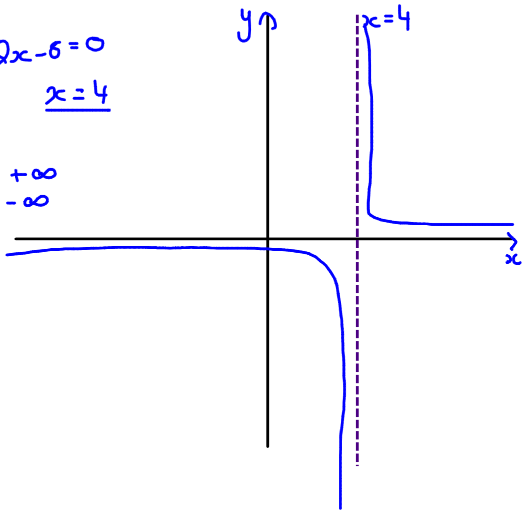
look at values close to $x=4$ e.g. 4.001 $y \to +\infty$

3.999 $y \to -\infty$

and very large x

as $x \to +\infty$ $y \to {}^{+}0$

$x \to -\infty$ $y \to -0$



Vertical Asymptotes of Rational Functions

2. $f(x)=\dfrac{5}{x^2+3x-4}$

$x^2+3x-4=0$ at vertical asymptote

$(x+4)(x-1)=0$

$\underline{x=-4}$, $\underline{x=1}$

As $x \to +\infty$, $y \to 0$

$x \to -\infty$, $y \to 0$

Values very close to $x=-4$

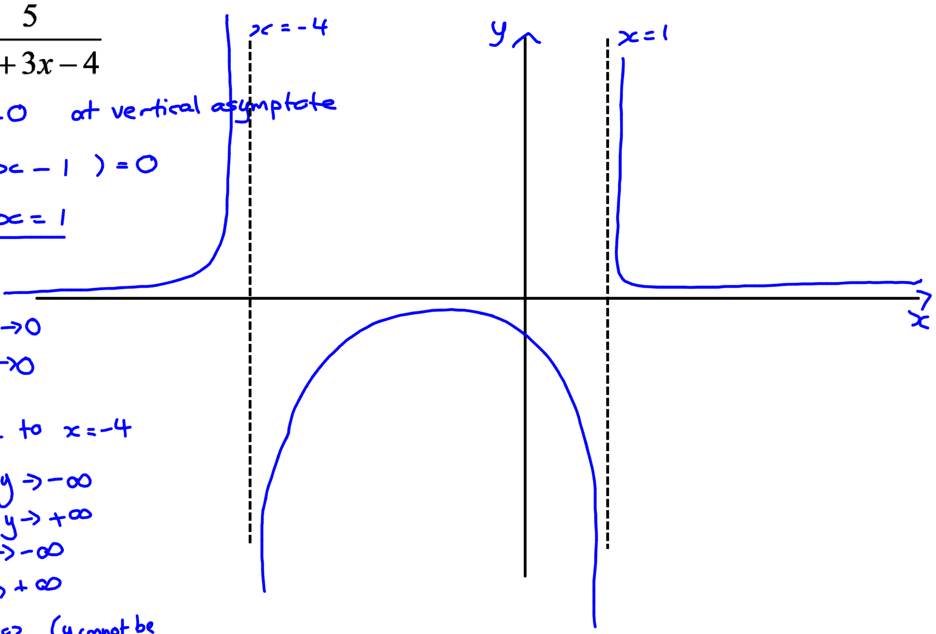
$x = -3.9999$, $y \to -\infty$

$x = -4.0001$, $y \to +\infty$

$x = 0.9999$, $y \to -\infty$

$x = 1.0001$, $y \to +\infty$

$x = -2$, $y = -0.83$ (y cannot be zero)



Non-vertical asymptotes of rational functions

◇ Non-vertical asymptotes can be horizontal asymptotes (have equations of the form y = a) or slant/oblique asymptotes (have equations of the form y = mx + c).

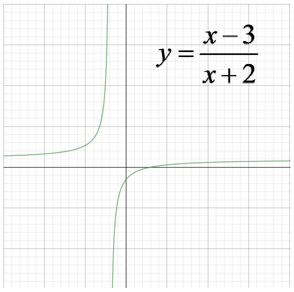
We can quickly spot when the function has a horizontal or oblique asymptote.

- Horizontal asymptotes: Numerator has same degree as denominator.
- Slant asymptotes: Degree of the numerator > degree of denominator.
- Asymptote is x-axis when Degree of numerator < degree of denominator

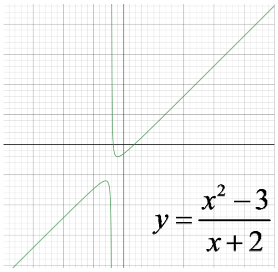
◇ When working with rational functions which involve improper fractions you MUST carry out algebraic long division to help identify the non-vertical asymptotes.

- Try to think what will happen to y when you get very large x!

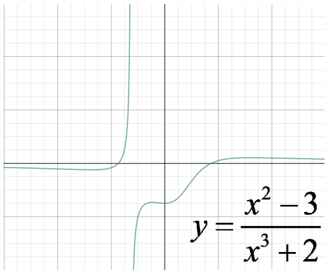
Non-vertical asymptotes of rational functions



$$y=\frac{x-3}{x+2}$$



$$y=\frac{x^2-3}{x+2}$$



$$y=\frac{x^2-3}{x^3+2}$$

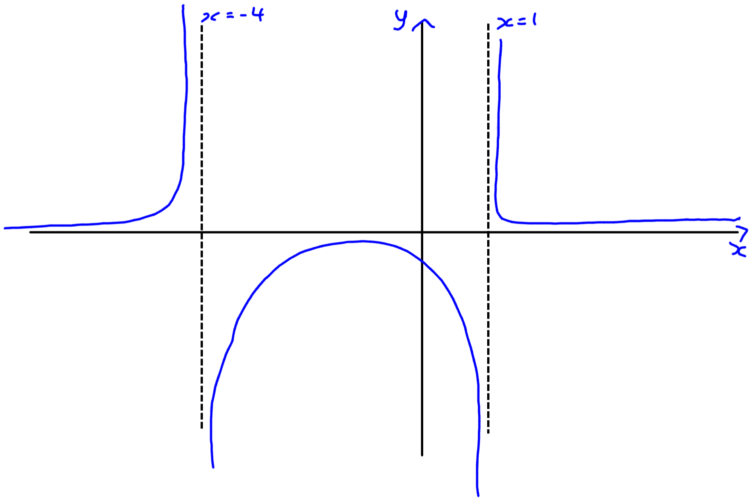
Non-vertical asymptotes of rational functions

Examples:

1. $y=\dfrac{5}{x^2+3x-4}$

Same as example ② Vertical asymptotes

degree of numerator $<$ degree of denominator $\Rightarrow$ x-axis is horizontal asymptote or $\underline{\underline{y=0}}$



Non-vertical asymptotes of rational functions

Examples:

2. $y = \dfrac{3x^2 - 4x + 2}{x^2 - 2x + 1}$

Vertical asymptotes :

$x^2-2x+1=0$

$(x-1)(x-1)=0$

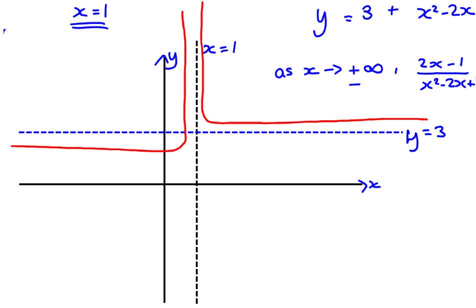
$\underline{\underline{x=1}}$

Non-Vertical :

$$\begin{array}{r|l} & \quad 3 \\ \hline x^2-2x+1 & 3x^2-4x+2 \\ & \ominus 3x^2 \overset{\ominus}{-}6x \overset{\ominus}{+}3 \\ \hline & \qquad 2x-1 \end{array}$$

$y = 3 + \dfrac{2x-1}{x^2-2x+1}$

as $x \to \pm\infty$, $\dfrac{2x-1}{x^2-2x+1} \to 0$ therefore $y \to 3$



values of x close to 1

$x=0.9999$, $y \to +\infty$ $x \to \pm\infty$, $y \to 3$

$x=1.0001$, $y \to +\infty$

There can be intersection points with non-vertical asymptotes E.g.

When $y=3$

$3 = 3 + \dfrac{2x-1}{x^2-2x+1}$

$0 = \dfrac{2x-1}{x^2-2x+1} \Rightarrow 2x-1=0$

$x=\frac{1}{2}$ $(\frac{1}{2},3)$

Try another value for x

e.g let $x=-2$

$y = 2.61...$

Non-vertical asymptotes of rational functions

Examples:

3. $y = \dfrac{2x^2 + x - 3}{x + 1}$

Vertical asymptote :

$x+1=0$

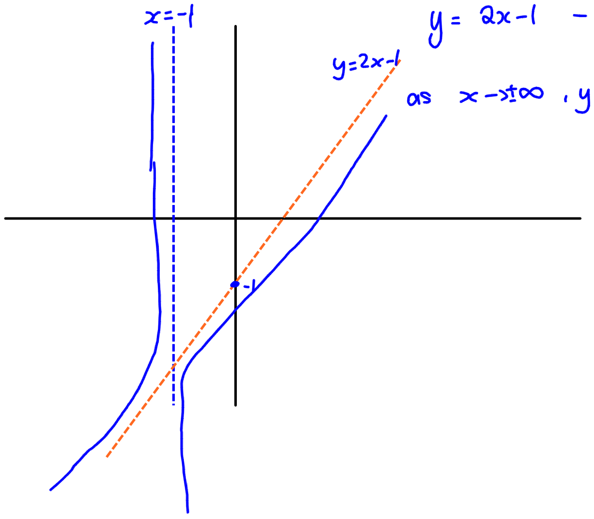
$x=-1$

Non-vertical :

$$\begin{array}{r|l} & \quad 2x-1 \\ \hline x+1 & 2x^2+x-3 \\ & \ominus 2x^2 \overset{\ominus}{+} 2x \\ \hline & \qquad -x-3 \\ & \qquad \ominus -x \overset{\ominus}{-} 1 \\ \hline & \qquad\quad -2 \end{array}$$

$y = 2x-1 - \dfrac{2}{x+1}$

as $x \to \pm\infty$, $y \to 2x-1$



values of x close to -1

$x=-1.0001$, $y \to +\infty$

$x=-0.999$, $y \to -\infty$

Past Paper Question (Specimen Paper)

13. Consider the curve in the (x, y) plane defined by the equation $y = \frac{4x-3}{x^2-2x-8}$.

(a) Identify the vertical asymptotes to this curve and justify your answer. **2**

Here are two statements about the curve:

(1) It does not cross or touch the x-axis.
(2) The line $y = 0$ is an asymptote.

(b) (i) State why statement (1) is false.

(ii) Show that statement (2) is true. **3**

(a) Vertical asymptotes are where the function is undefined. i.e. when denominator = 0

$\Rightarrow x^2-2x-8=0$ $\quad x \to 4, y \to \infty$

$(x-4)(x+2)=0$ $\quad x \to -2, y \to -\infty$

$\underline{x=4}, \underline{x=-2}$

(b) (i) If graph cuts x-axis, $y=0$

$\frac{4x-3}{x^2-2x8}=0 \Rightarrow 4x-3=0$ graph cuts x-axis at $(\frac{3}{4}, 0) \Rightarrow$ Statement ① is false.

$x=3/4$

(ii) $y=0$ is an asymptote

as $x \to +\infty, y \to +0$

$x \to -\infty, y \to -0$

Because the graph never reaches 0 but gets infinitely closer, $y=0$ is an asymptote. (Also the degree of the numerator < degree of denominator).

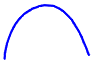
Critical Points

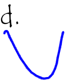
These are points where f′(x) = 0 or where f′(x) is undefined. They include max. and min. turning points, points of inflection, points where the function is discontinuous or at a corner and endpoint max. and min. within closed intervals too.

Concavity and Points of Inflection

If f''(x) < 0, a curve is concave downward.

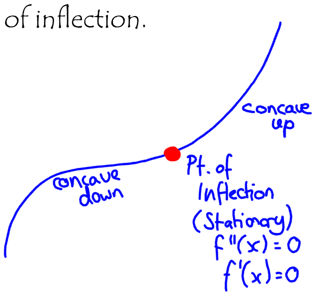
If f''(x) > 0, the curve is concave upward.

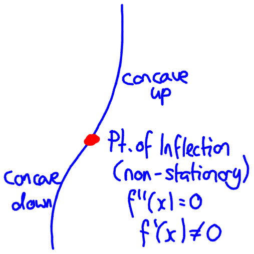


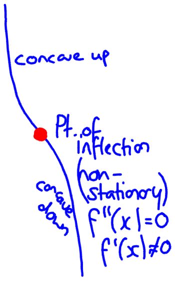


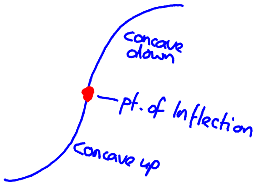
A point of inflection is where the curve changes from concave down to concave up or vice versa. They aren't necessarily stationary points.

At the point of inflection f''(x) = 0 but f'(x) = 0 only when it is a stationary point of inflection.









Points of Inflection

Example:

Find the point of inflection of the curve y = $x^3 - 3x^2 - 2$ and state whether it is stationary or not.

$y = x^3 - 3x^2 - 2$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{d^2y}{dx^2} = 6x - 6$ At point of inflection $\frac{d^2y}{dx^2} = 0$

$6x - 6 = 0$

$\underline{x = 1}$

$y = 1^3 - 3(1)^2 - 2$

$y = 1 - 3 - 2$

$\underline{\underline{y = -4}}$

$(1, -4)$

$\frac{dy}{dx}$ at $x = 1$: $3(1)^2 - 6(1)$

$= 3 - 6$

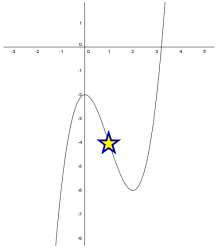
$= -3 \neq 0$

$\Rightarrow$ Not a stationary point of inflection.

Concavity – Check

x	0	1	2
$\frac{d^2y}{dx^2}$	<0	0	>0
concavity	∩		∪

Graph goes from concave down to concave up



The Modulus Function

The modulus (absolute value) of a real number is the value of the number regardless of the sign.

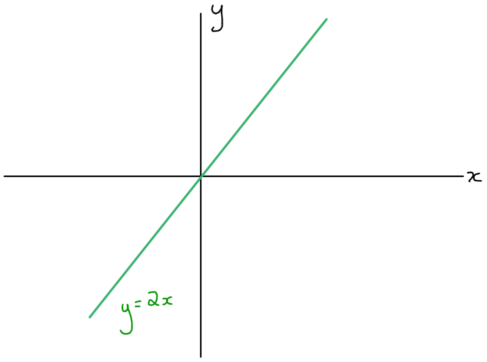
To draw the graph of the modulus y will always be positive, therefore reflect any part below the x- axis in the x-axis so that all the graph lies above the x-axis.

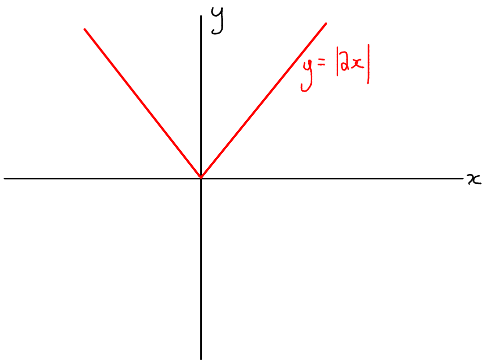
The modulus function is now the combined effect of the positive part of the original function and the new reflected part.

The Modulus Function

Examples:

1. Draw the graph of the function y = |2x|

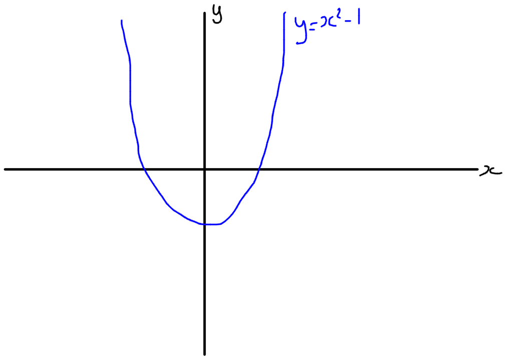


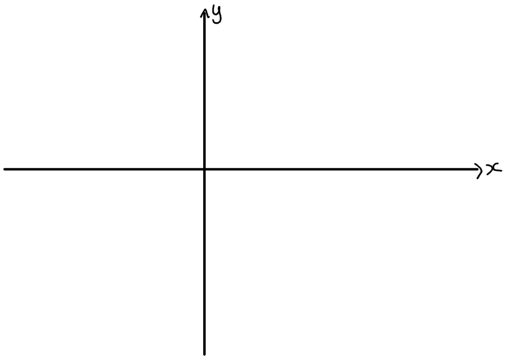


The Modulus Function

Examples:

2. Draw the graph of the function y = $|x^2 - 1|$

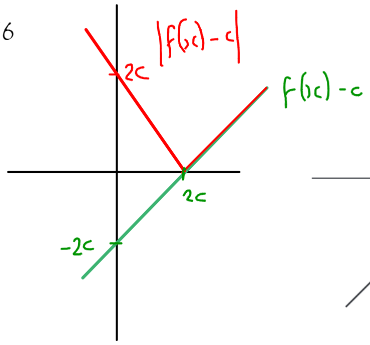




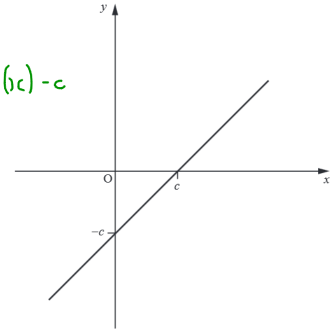
The Modulus Function

Past Paper Question:

2016



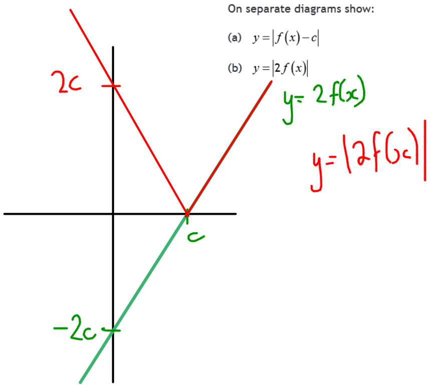
12. Below is a diagram showing the graph of a linear function, $y = f(x)$.



On separate diagrams show:

(a) $y = |f(x) - c|$ 2

(b) $y = |2f(x)|$ 2

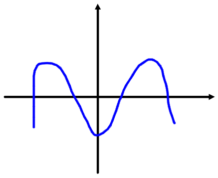


Odd and Even Functions

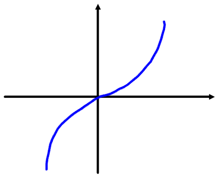
Recall: $y = -f(x)$ is a reflection in the x-axis

$y = f(-x)$ is a reflection in the y-axis

Given a function f, if $f(-x) = f(x)$ for all values of x, then it is said to be an even function. It will therefore be symmetrical about the y - axis.



Given a function f, if $f(-x) = -f(x)$ for all values of x, then f is said to be an odd function. The graph of an odd function will have half-turn symmetry about the origin.



To determine if an expression is odd, even or neither, find an expression for $f(-x)$ and compare it to $f(x)$.

Odd and Even Functions

Examples:

1. Show that $f(x) = 4x^3 - x$ is an odd function.

If odd, then $f(-x) = -f(x)$

$f(-x) = 4(-x)^3 - (-x) = -4x^3 + x$

$-f(x) = -(4x^3 - x) = -4x^3 + x$

$f(-x) = -f(x) \Rightarrow$ function is odd

2. Prove that the function $f(x) = x^4 - 2x^2 + 3$ is an even function.

If even, then $f(-x) = f(x)$

$f(-x) = (-x)^4 - 2(-x)^2 + 3$

$= x^4 - 2x^2 + 3 = f(x)$

$\Rightarrow$ function is even

Odd and Even Functions

Examples:

3. Prove that f(x) = x^3 – 2x is an odd function.

4. Investigate whether f(x) = x^3sinx is odd, even or neither.

$f(-x) = (-x)^3 \sin(-x)$

$= -x^3(-\sin x)$

$= x^3 \sin x \quad = f(x)$

$\sin(-x) = -\sin x$

$\Rightarrow$ function is even for all values of x

Odd and Even Functions

Past Paper Questions:

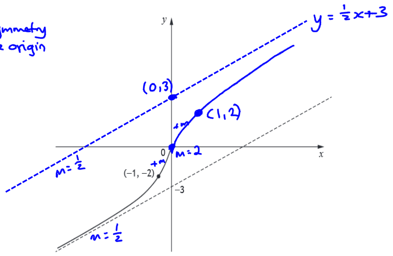
2017

(a)

Odd function $\Rightarrow$ $\frac{1}{2}$ turn symmetry about the origin

12. In the diagram below part of the graph of $y = f(x)$ has been omitted.

The point (–1, –2) lies on the graph and the line $y = \frac{1}{2}x - 3$ is an asymptote.



Given that $f(x)$ is an odd function:

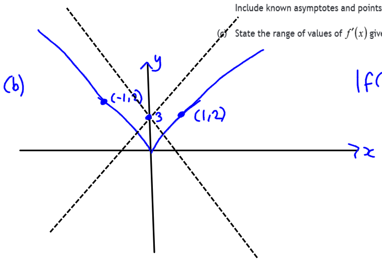
(a) Copy and complete the diagram, including any asymptotes and any points you know to be on the graph. 2

(b) $g(x) = |f(x)|$. On a separate diagram, sketch $g(x)$.

Include known asymptotes and points. 2

(c) State the range of values of $f'(x)$ given that $f'(0) = 2$. 1

(b)



$|f(x)| = g(x)$

(c) $f'(0) = 2$

At the origin the gradient = 2

The gradient of the asymptotes are = $\frac{1}{2}$

therefore

$\frac{1}{2} < m \leq 2$

so $\underline{\frac{1}{2} < f'(x) \leq 2}$

Odd and Even Functions

Past Paper Questions:

2015

Marks

14. For some function, f, define

$$g(x) = f(x) + f(-x) \quad \text{and}$$
$$h(x) = f(x) - f(-x).$$

Show that $g(x)$ is an even function and that $h(x)$ is an odd function.

Hence show that $f(x)$ can be expressed as the sum of an even and an odd function. **4**

A function is even if $f(x) = f(-x)$

so $g(x)$ is even if $g(x) = g(-x)$

$g(x) = f(x) + f(-x)$

$g(-x) = f(-x) + f(-(-x))$

$= f(-x) + f(x) = g(x)$ $\therefore$ even

$h(x) = f(x) - f(-x)$ A function is odd if $-f(x) = f(-x)$

$h(-x) = f(-x) - f(-(-x))$

$= f(-x) - f(x)$

$-h(x) = -f(x) + f(-x) = h(-x)$ $\therefore$ Function is odd

Hence $g(x) = f(x) + f(-x)$

$h(x) = f(x) - f(-x)$

$f(-x) = g(x) - f(x)$

$f(-x) = f(x) - h(x)$

$g(x) - f(x) = f(x) - h(x)$

$2f(x) = h(x) + g(x)$

$f(x) = \frac{1}{2}\left[h(x) + g(x)\right]$

Since $g(x)$ even and $h(x)$ odd,

$f(x)$ is the sum of even and odd functions.

Graph Sketching

- Find roots and y - intercept.
- Determine vertical and non-vertical asymptotes.
- Investigate how the graph approaches the asymptotes.
- Use differentiation if there are S.P.'s to consider.

Graph Sketching

Examples:

1. $y = \frac{4}{x-2}$

Look for asymptotes

Vertical $\Rightarrow x-2=0$

$x=2$

x-axis is an asymptote

$x \to +\infty \quad y \to 0^+$

$x \to -\infty \quad y \to 0^-$

values close to 2

$x \to 2 \quad , y \to \pm\infty$

S.P's $\frac{dy}{dx} = 0 \qquad y = 4(x-2)^{-1}$

$\frac{dy}{dx} = -4(x-2)^{-2}(1)$

$= \frac{-4}{(x-2)^2} \neq 0$

no S.P's

Cuts x-axis $\Rightarrow y=0$

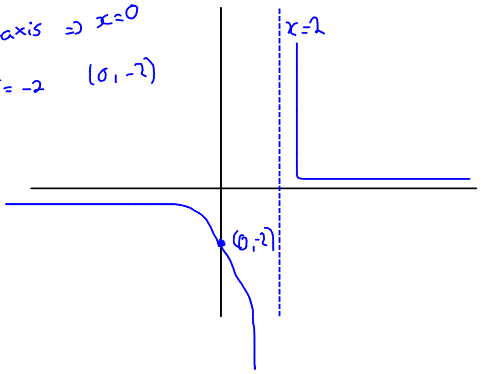
$\frac{4}{x-2} \neq 0$

cuts y-axis $\Rightarrow x=0$

$\frac{4}{0-2} = -2 \qquad (0,-2)$

$\frac{-4}{(x-2)^2} < 0$ always

$\Rightarrow m < 0$



Graph Sketching

Examples:

2. $f(x) = \frac{2x+5}{1-x}$

Asymptotes

Vertical : $1-x=0$

$x=1$

Values close to 1

$x = 1.0001 \ , \ y = -70002$

$x = 0.999 \ , \ y = 6998$

Horizontal

$$\begin{array}{r} -2 \\ -x+1 \overline{) \ 2x+5} \\ \ominus 2x \ominus 2 \\ \hline 7 \end{array} \qquad y = -2 + \frac{7}{1-x}$$

Horizontal asymptote at $y = -2$

as $x \to +\infty \ , \ y \to -2$

$x \to -\infty \ , \ y \to -2$

x-intercepts $\Rightarrow y=0$

$-2 + \frac{7}{1-x} = 0$

$\frac{7}{1-x} = 2$

$7 = 2-2x$

$5 = -2x$

$x = -2.5 \quad (-2.5, 0)$

y-intercept $\Rightarrow x=0$

$-2 + \frac{7}{1} = y$

$y = 5 \quad (0,5)$

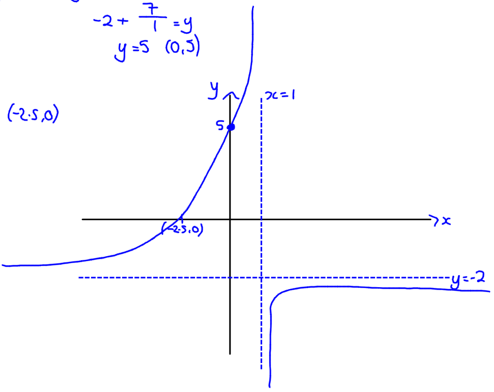
S.P.'s

$f'(x) = \frac{(1-x)(2) - (2x+5)(-1)}{(1-x)^2}$

$= \frac{2-2x+2x+5}{(1-x)^2}$

$= \frac{7}{(1-x)^2} > 0$ always

$\Rightarrow$ no s.p's



Graph Sketching

Examples:

3. $f(x) = \dfrac{2x+5}{x^2+5x+4}$

Asymptotes:

Horizontal at $y=0$

$x \to +\infty, y \to {}^{+}0$

$x \to -\infty$ $y \to {}^{-}0$

Vertical

$x^2+5x+4=0$

$(x+1)(x+4)=0$

$x=-1, -4$

close to -1

$x \to^{-} -1, y \to -\infty$

$x \to^{+} -1, y \to +\infty$

close to -4

$x \to^{-} -4, y \to -\infty$

$x \to^{+} -4, y \to +\infty$

x - intercepts

$y=0 \Rightarrow 2x+5=0$

$x = \frac{-5}{2}$ $(\frac{-5}{2}, 0)$

y - intercept

$x=0 \Rightarrow \frac{5}{4} = (0, 1.25)$