Parametric Equations

A parameter is a quantity that influences the behaviour of a function.

They are like variables but are held constant for use in a function.

Varying the parameter gives you a family of functions.

We can express both x and y in terms of a parameter.

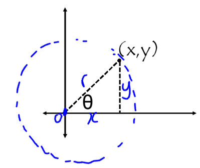
The parameter we tend to use is t for Cartesian functions and θ for trigonometric functions.

E.g. $x = 3t + 2$ and $y = t^3 + 1$

Parametric Equations

We can express the equation of the circle in parametric form.

$x^2+y^2=r^2$



$\cos\theta = \frac{x}{r}$ $\quad \sin\theta = \frac{y}{r}$

$\boxed{x = r\cos\theta \quad y = r\sin\theta}$

$x^2 = r^2\cos^2\theta \quad y^2 = r^2\sin^2\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta)$

$x^2+y^2=r^2$

Parametric Equations

The Cartesian equation can be found by substituting one equation into the other to eliminate t. This equation is known as the **constraint** equation.

Example: Express the parametric equations $x = \frac{3+t}{2-t}$ $\quad y = \frac{1-3t}{2-t}$ $\quad , t \neq 2$

in Cartesian from and show that the equation represents a line.

$2x - xt = 3 + t$ $\qquad 2y - yt = 1 - 3t$

$2x - 3 = t + tx$ $\qquad 2y - 1 = -3t + yt$

$2x - 3 = t(1+x)$ $\qquad 2y - 1 = t(y-3)$

$t = \frac{2x-3}{1+x}$ $\qquad t = \frac{2y-1}{y-3}$

$\frac{2x-3}{1+x} = \frac{2y-1}{y-3}$

$(2x-3)(y-3) = (2y-1)(1+x)$

$\cancel{2xy} - 6x - 3y + 9 = 2y + \cancel{2xy} - 1 - x$

$-5x - 5y + 10 = 0$

$x + y - 2 = 0$ in the form $Ax+By+C=0$, ∴ equation of a line

Parametric Differentiation

We use the chain rule to differentiate parametric functions.

$\frac{dy}{dx} = \boxed{\frac{dy}{dt} \div \frac{dx}{dt}}$

$\frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dx}$

Examples: Find $\frac{dy}{dx}$

1. $x = 4 + 4t$ and $y = 3 - 3t^2$

$\frac{dx}{dt} = 4 \qquad \frac{dy}{dt} = -6t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-6t}{4} = \frac{-3t}{2}$

Parametric Differentiation

Examples: Find $\frac{dy}{dx}$

2. $x = t^2 + \frac{1}{t^2} \qquad y = t^2 - \frac{1}{t^2}$

$x = t^2 + t^{-2} \qquad y = t^2 - t^{-2}$

$\frac{dx}{dt} = 2t - 2t^{-3} = 2t - \frac{2}{t^3} \qquad \frac{dy}{dt} = 2t + 2t^{-3} = 2t + \frac{2}{t^3}$

$\frac{dy}{dx} = \frac{2t + \frac{2}{t^3}}{2t - \frac{2}{t^3}} = \frac{2t^4+2}{2t^4-2} = \frac{2(t^4+1)}{2(t^4-1)} = \frac{t^4+1}{t^4-1}$

Parametric Differentiation

Examples: Find $\frac{dy}{dx}$

3. $x = \theta - \sin\theta \qquad y = 1 - \cos\theta \qquad (0 \le \theta < 2\pi)$

$\frac{dx}{d\theta} = 1 - \cos\theta \qquad \frac{dy}{d\theta} = \sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}$

Parametric Differentiation

Examples: Find $\frac{dy}{dx}$ in terms of x

4. $x = \sqrt{t}$ $\quad y = t^2 - 3$

$x = t^{\frac{1}{2}}$

$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$ $\quad dy/dt = 2t$

$= \frac{1}{2t^{\frac{1}{2}}}$

$dy/dx = 2t \div \frac{1}{2t^{\frac{1}{2}}}$

$= 2t \times 2t^{\frac{1}{2}}$

$= \underline{\underline{4t^{\frac{3}{2}}}}$ $\quad x = \sqrt{t}$, $x^2 = t$

$dy/dx = 4(x^2)^{\frac{3}{2}} = \underline{\underline{4x^3}}$

Parametric Differentiation

Examples:

5. A curve is defined by the parametric equations

$$x = t^2 + 1 \qquad y = t(t^2 + 1)$$

for all t. Find the equation of the tangent to the curve at the point with parameter t = 2.

$x = t^2 + 1$ $\quad y = t^3 + t$

$\frac{dx}{dt} = 2t$ $\quad dy/dt = 3t^2 + 1$

$dy/dx = \frac{3t^2+1}{2t}$ at $t = 2 \Rightarrow \frac{3(2)^2+1}{2(2)} = \underline{\underline{\frac{13}{4}}} = m$

$x = t^2 + 1$

$x = 2^2 + 1 = 5$

$y = t^3 + t$

$y = 2^3 + 2 = 10$ $\quad \underline{(5, 10)}$

$y - 10 = \frac{13}{4}(x - 5)$

$4y - 40 = 13x - 65$

$\underline{13x - 4y - 25 = 0}$

Parametric Differentiation

6. Given that x = 2sinθ and y = 1 – cos2θ (0 ≤ θ ≤ π/2)

show that

$$\frac{dy}{dx} = 2\sin\theta$$

$\frac{dx}{d\theta} = 2\cos\theta$ $\quad dy/d\theta = 2\sin 2\theta$

$dy/dx = \frac{2\sin 2\theta}{2\cos\theta} = \frac{\sin 2\theta}{\cos\theta} = \frac{2\sin\theta\cos\theta}{\cos\theta} = \underline{\underline{2\sin\theta}}$ as required

Parametric Differentiation (The second derivative)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$= \frac{d(dy/dx)}{dx}$ $\quad \div dt$ $\quad \div dt$ $\quad = \frac{\frac{d}{dt}(dy/dx)}{\frac{dx}{dt}}$

$d^2y/dx^2 = \underline{\underline{\frac{d}{dt}(dy/dx) . \frac{dt}{dx}}}$

Parametric Differentiation (The second derivative)

Examples:

1. Find $\frac{d^2y}{dx^2}$ when x = t² and y = 2t

$\frac{dx}{dt} = 2t$ $\quad \frac{dy}{dt} = 2$

$$\boxed{\frac{1}{dx/dt} = dt/dx}$$

$dy/dx = \frac{2}{2t} = \frac{1}{t} = t^{-1}$

$d^2y/dx^2 = d/dt(dy/dx) . \frac{1}{dx/dt}$

$= (-t^{-2}) . \frac{1}{2t}$

$= -\underline{\underline{\frac{1}{2t^3}}}$

Parametric Differentiation (The second derivative)

Examples:

2. A curve is defined by the parametric equations

$$x = \frac{2t}{1-t^2} \qquad y = \frac{1+t^2}{1-t^2}$$

Show that $\frac{dy}{dx} = \frac{2t}{1+t^2}$ and $\frac{d^2y}{dx^2} = \left(\frac{1-t^2}{1+t^2}\right)^3$

$\frac{dx}{dt} = \frac{(1-t^2).2 - 2t(-2t)}{(1-t^2)^2}$ $\quad \frac{dy}{dt} = \frac{(1-t^2)(2t) - (1+t^2)(-2t)}{(1-t^2)^2}$

$= \frac{2-2t^2+4t^2}{(1-t^2)^2}$ $\quad = \frac{2t-2t^3+2t+2t^3}{(1-t^2)^2}$

$= \frac{2+2t^2}{(1-t^2)^2}$ $\quad = \frac{4t}{(1-t^2)^2}$

$dy/dx = \frac{4t}{(1-t^2)^2} \div \frac{2+2t^2}{(1-t^2)^2} = \frac{4t}{(1-t^2)^2} \times \frac{(1-t^2)^2}{2+2t^2} = \frac{4t}{2(1+t^2)} = \underline{\underline{\frac{2t}{1+t^2}}}$ as required

$d^2y/dx^2 \Rightarrow \frac{(1+t^2)(2)-2t(2t)}{(1+t^2)^2} . \frac{1}{\frac{2+2t^2}{(1-t^2)^2}} =$

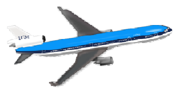
$\frac{2+2t^2-4t^2}{(1+t^2)^2} . \frac{(1-t^2)^2}{2+2t^2} = \frac{\cancel{2}(1-t^2)(1-t^2)^2}{\cancel{2}(1+t^2)^2(1+t^2)} = \underline{\underline{\frac{(1-t^2)^3}{(1+t^2)^3}}}$ as required

Motion in a plane

Car stays on ground.

Moves along one axis

with respect to time.



Plane moves forward as well as up or down.

Moves along both axes

with respect to time.

Motion in a plane

Until now, we have looked at rectillinear motion (in a straight line)

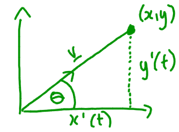
Particles are more likely to move in a plane.

Given an xy plane we let:

x be the horizontal component

y be the vertical component

t be a parameter to represent the time.



The horizontal velocity would then be $\frac{dx}{dt}$ or $\dot{x}$

The vertical velocity would be $\frac{dy}{dt}$ or $\dot{y}$

$$\text{Speed} = |v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The direction of motion at any instant of time is $\frac{dy}{dx} = \tan\theta$

$\frac{dy/dt}{dx/dt}$

Motion in a plane

We can then say that acceleration is

$$\sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

Motion in a plane

Examples:

1. A particle is moving along a path determined by the parametric equations

$x = 4t - 1$ $\quad\quad$ $y = t^2 + 3t$

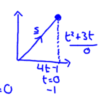
where t represents time in seconds and distance is measured in metres.

(a) How far is the particle from the origin at t = 0?

(b) When t = 2, calculate the speed of the particle

(c) Calculate the direction the particle is moving in when t = 2

(a)



$t=0$

$|s| = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$ metre from origin.

(b)

$dy/dt = 2t+3$

at $t=2$

$dy/dt = \underline{7}$

speed $= |v| = \sqrt{(4)^2 + (7)^2}$

$= \sqrt{65} = \underline{\underline{8.1\,ms^{-1}}}$

(c) Direction at $t=2$

$\tan\theta = dy/dx = \frac{7}{4}$

$\theta = \tan^{-1}\left(\frac{7}{4}\right) = 60^\circ$ (Rounded)

Bearing $= 90^\circ - 60^\circ = \underline{\underline{030^\circ}}$

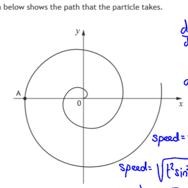
Motion in a plane

2. The position of a particle at time t is given by the parametric equations $\quad$ 2017 PP Q

$x = t\cos t, \quad y = t\sin t, \quad t \geq 0.$

(a) Find an expression for the instantaneous speed of the particle. **5**

The diagram below shows the path that the particle takes.



$\frac{dx}{dt} = t(-\sin t) + \cos t$

$= -t\sin t + \cos t$

$dy/dt = t(\cos t) + \sin t$

$= t\cos t + \sin t$

$\text{speed} = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2}$

$\text{speed} = \sqrt{t^2\sin^2 t - \cancel{2t\sin t\cos t} + \cos^2 t + t^2\cos^2 t + \cancel{2t\cos t\sin t} + \sin^2 t}$

$\text{speed} = \sqrt{t^2(\sin^2 t + \cos^2 t) + 1}$

$\text{Speed} = \underline{\underline{\sqrt{t^2+1}}}$

(b) Calculate the instantaneous speed of the particle at point A. **2**

At point A, $y = 0$

$y = t\sin t$

$0 = t\sin t$

$t = 0$ or $\sin t = 0$

$t = 0, \pi, 2\pi, 3\pi \ldots$

Look for a t that gives a negative x

$x = t\cos t$

at $t = 3\pi$ $x = -9.42\ldots$

$A(-9.42\ldots, 0)$

Speed at point A $= \sqrt{(3\pi)^2 + 1} = \underline{\underline{\sqrt{9\pi^2+1}}}$ or $\underline{\underline{9.48}}$ (3 s.f.)