Starter 17.8.2016

Use the binomial theorem to expand and simplify

$$\left(\frac{x^2}{3}-\frac{2}{x}\right)^5.$$

$$= \left(\tfrac{1}{3}x^2 - 2x^{-1}\right)^5 \qquad (a+b)^n = \sum_{n=0}^{r} \binom{n}{r} a^{n-r} b^r \quad 4$$

$$= \binom{5}{0}\left(\tfrac{1}{3}x^2\right)^5 + \binom{5}{1}\left(\tfrac{1}{3}x^2\right)^4\left(-2x^{-1}\right) + \binom{5}{2}\left(\tfrac{1}{3}x^2\right)^3\left(-2x^{-1}\right)^2$$

$$+ \binom{5}{3}\left(\tfrac{1}{3}x^2\right)^2\left(-2x^{-1}\right)^3 + \binom{5}{4}\left(\tfrac{1}{3}x^2\right)\left(-2x^{-1}\right)^4$$

$$+ \binom{5}{5}\left(-2x^{-1}\right)^5$$

Today we will be learning how to write a rational function as partial fractions.

Partial Fractions

Writing a function as partial fractions is reversing the process of writing a sum as a single fraction.

$$\frac{3}{x+5} + \frac{4}{x} = \frac{7x+20}{x(x+5)}$$

How can you reverse the process?

Try this one $\quad \dfrac{x+2}{(x+4)(x+5)}$

Partial Fractions

The fractions must be proper i.e. the degree of the polynomial in the numerator must be smaller than the degree of the polynomial in the denominator.

The denominator must be in its factorised form so you can easily identify how to split it up.

Partial Fractions

Type 1: Denominator has distinct linear factors.

Examples: Express as partial fractions

1. $\dfrac{3x}{x^2+3x+2} = \dfrac{3x}{(x+2)(x+1)}$

$$= \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

$$\frac{A(x+1)+B(x+2)}{(x+1)(x+2)} = \frac{Ax+A+Bx+2B}{(x+1)(x+2)}$$

Equate like terms

$Ax+Bx=3x \qquad A+2B=0$

$\ominus$ $A+B=3$

$A+2B=0$

$-B=3$

$B=3$

$A+B=3$

$A-3=3$

$A=6$

$$\frac{6}{x+2} - \frac{3}{x+1}$$

Partial Fractions

2. $\dfrac{5x-4}{x^2-x-2} = \dfrac{5x-4}{(x-2)(x+1)}$

Ex 2.2 pg. 23

$$= \frac{A}{(x-2)} + \frac{B}{(x+1)} = \frac{Ax+A+Bx-2B}{(x-2)(x+1)}$$

$Ax+Bx=5 \qquad A-2B=-4$

$\ominus$ $A+B=5$

$A-2B=-4$

$3B=9$

$B=3$

$A-2(3)=-4$

$A-6=-4$

$A=2$

$$\frac{2}{x-2} + \frac{3}{x+1}$$

Partial Fractions

Type 2: A denominator with a repeated factor.

Example: $\frac{3x+7}{(x+1)^2}$ $\quad \frac{a}{x+1} + \frac{b}{(x+1)^2}$

Think about how you would split this one.

$$\frac{a(x+1)+b}{(x+1)^2} = \frac{ax+a+b}{(x+1)^2}$$

$a=3 \quad a+b=7$

$b=4$

$$\frac{3}{x+1} + \frac{4}{(x+1)^2}$$

Partial Fractions

Example 2:

$$\frac{4x}{x^2-6x+9} = \frac{4x}{(x-3)(x-3)} = \frac{4x}{(x-3)^2}$$

$$\frac{a}{x-3} + \frac{b}{(x-3)^2}$$

$$\frac{a(x-3)+b}{(x-3)^2} = \frac{ax-3a+b}{(x-3)^2}$$

$a=4$

$-3a+b=0$

$-12+b=0$

$b=12$

$$\frac{4}{x-3} + \frac{12}{(x-3)^2}$$

Partial Fractions

Exercise

Express each single fraction in partial fractions.

1) $\frac{3x+1}{(x+1)^2}$ 2) $\frac{2x-7}{(x-2)^2}$ 3) $\frac{x-3}{(x-1)^2}$ 4) $\frac{2x+11}{(x+4)^2}$

Partial Fractions

Type 3: There are 3 factors in the denominator (to give a cubic). These can include a repeated factor too.

Examples:

1. Express $\frac{5x^2-20x+17}{(x-2)(2x-5)(x+1)}$ in partial fractions.

$$= \frac{a}{(x-2)} + \frac{b}{(2x-5)} + \frac{c}{(x+1)}$$

$$\frac{a(2x-5)(x+1)+b(x-2)(x+1)+c(x-2)(2x-5)}{(x-2)(2x-5)(x+1)}$$

$$\frac{a[2x^2-3x-5]+b[x^2-x-2]+c[2x^2-9x+10]}{(x-2)(2x-5)(x+1)}$$

numerator

$\to 2ax^2-3ax-5a+bx^2-bx-2b+2cx^2-9cx+10c$

$x^2(2a+b+2c)+x(-3a-b-9c)-5a-2b+10c$

$2a+b+2c=5$ $\quad -3a-b-9c=-20 \quad -5a-2b+10c=17$

$b=-2a-2c+5$

$-3a-(-2a-2c+5)-9c=-20$

$-5a-2(-2a-2c+5)+10c=17$

$-a-7c=-15$

$-a+14c=27$

$-21c=-42$

$c=2$

$-a-7(2)=-15$

$-a-14=-15$

$-a=-1$

$a=1$

$b=-2(1)-2(2)+5$

$b=-1$

$$\frac{1}{x-2} - \frac{1}{2x-5} + \frac{2}{x+1}$$

Partial Fractions

2. $\frac{x^2+11x+15}{(x-1)(x+2)^2}$ $\quad \frac{a}{(x-1)} + \frac{b}{(x+2)} + \frac{c}{(x+2)^2}$

$$\frac{a(x+2)^2+b(x-1)(x+2)+c(x-1)}{(x-1)(x+2)^2}$$

$= a[x^2+4x+4]+b[x^2+x-2]+cx-c$ numerator

$ax^2+4ax+4a+bx^2+bx-2b+cx-c$

$x^2[a+b]+x[4a+b+c]+4a-c-2b$

$a+b=1 \quad 4a+b+c=11 \quad 4a-c-2b=15$

$a=1-b$

$4(1-b)+b+c=11$

$4-4b+b+c=11$

$-3b+c=7$

$4(1-b)-c-2b=15$

$4-4b-c-2b=15$

$-6b-c=11$

$-3b+c=7$

$+ \; -6b-c=11$

$-9b=18$

$b=-2$

$a=1-b$

$a=1-(-2)$

$a=3$

$-6(-2)-c=11$

$12-c=11$

$c=1$

$$\frac{3}{(x-1)} - \frac{2}{(x+2)} + \frac{1}{(x+2)^2}$$

Partial Fractions

3. $\frac{x^2+11x+15}{x^3+3x^2-4}$

x^3+3x^2-4 (need to factorise)

$x=1$ is a root

1	1	3	0	-4
		1	4	4
	1	4	4	$\boxed{0}$

$(x-1)(x^2+4x+4)$

$(x-1)(x+2)(x+2)$

$$\frac{x^2+11x+15}{(x-1)(x+2)^2}$$

$$\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Same as previous example!

$$\frac{3}{x-1} - \frac{2}{x+2} + \frac{1}{(x+1)^2}$$

Partial Fractions

4. $\dfrac{x^2 + 5x - 3}{(x-2)^3} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3}$

$$\frac{A(x-2)^2 + B(x-2) + C}{(x-2)^3}$$

$$\frac{A(x^2-4x+4) + Bx - 2B + C}{(x-2)^3} = \frac{Ax^2 - 4Ax + 4A + Bx - 2B + C}{(x-2)^3}$$

$A = 1$ $\quad -4A + B = 5$ $\quad 4A - 2B + C = -3$

$-4 + B = 5$ $\quad 4 - 18 + C = -3$

$B = 9$ $\quad C = 11$

$$\frac{1}{(x-2)} + \frac{9}{(x-2)^2} + \frac{11}{(x-2)^3}$$

Partial Fractions

Type 4: A denominator with an irreducible quadratic factor.

In this case the denominator cannot be fully factorised. (use the discriminant to check/prove it is irreducible)

We write these in the form $\dfrac{A}{x+p} + \dfrac{Bx+C}{q(x)}$ where q(x) represents the quadratic factor.

Examples: Express in partial fractions

1. $\dfrac{5x+1}{(x-1)(x^2+x+1)}$

$x^2 + x + 1$
$a=1 \quad b=1 \quad c=1$
$b^2 - 4ac$
$1^2 - 4(1)(1)$
$= 1 - 4$
$= -3 < 0 \Rightarrow$ Irreducible quadratic factor

$\Rightarrow \dfrac{A}{(x-1)} + \dfrac{Bx+C}{x^2+x+1}$

$$\frac{A(x^2+x+1) + (Bx+C)(x-1)}{(x-1)(x^2+x+1)} = \frac{Ax^2 + Ax + A + Bx^2 - Bx + Cx - C}{(x-1)(x^2+x+1)}$$

$x^2(A+B) + x(A-B+C) + A - C = 5x + 1$

$A + B = 0 \quad A - B + C = 5 \quad A - C = 1$

$A = -B \quad -B - B + C = 5 \quad -B - C = 1$

$-2B + C = 5$

$-B - C = 1$

$-3B = 6$

$A = -(-2) \quad B = -2 \quad -(-2) - C = 1$

$A = 2 \quad 2 - C = 1$

$C = 1$

$$\frac{2}{(x-1)} + \frac{1-2x}{(x^2+x+1)}$$

Examples: Express in partial fractions

2. $\dfrac{5}{x^3 - 2x^2 + x - 2} = \dfrac{5}{(x-2)(x^2+1)} = \dfrac{a}{x-2} + \dfrac{Bx+c}{x^2+1}$

$ax^2 + a + Bx^2 - 2Bx + cx - 2c$

$x^2(a+B) + x(-2B+c) + a - 2c$

$a + B = 0 \quad -2B + C = 0 \quad a - 2c = 5$

$a = -B \quad -2B + C = 0 \ (\times 2) \quad -4B + 2C = 0$

$-B - 2c = 5 \quad -B - 2C = 5$

$a = -(-1) \quad -(-1) - 2c = 5 \quad -5B = 5$

$a = 1 \quad 1 - 2c = 5 \quad B = -1$

$-2c = 4$

$c = -2$

$$\frac{1}{(x-2)} + \frac{-x-2}{x^2+1}$$

Today we will be learning how to carry out algebraic long division.

Algebraic Long Division

Algebraic long division can be used to factorise a polynomial.

We can also use it when we have been given a fraction where the numerator has a higher power of x than the denominator i.e. improper fractions.

$\dfrac{23260}{15}$

$$\begin{array}{r} 1550 \\ 15\overline{)23260} \\ -15 \\ \hline 82 \\ -75 \\ \hline 76 \\ -75 \\ \hline 10 \end{array}$$

$155\frac{10}{15}$

Algebraic Long Division

Examples:

1. $x^2 + 2x + 4 \div (x + 1)$

$$\begin{array}{r} x + 1 \\ x+1\overline{)x^2 + 2x + 4} \\ -(x^2 + 1x) \\ \hline x + 4 \\ -(x + 1) \\ \hline 3 \end{array}$$

$\frac{22}{7} = 3\frac{1}{7}$

$$x^2 + 2x + 4 \div (x+1) = x + 1 + \frac{3}{x+1}$$

Algebraic Long Division

Examples:

2. $x^3 + 4x^2 - x + 2 \div (x^2 + x)$

$$\begin{array}{r} x + 3 \\ x^2+x \overline{) x^3 + 4x^2 - x + 2} \\ -(x^3 + x^2) \\ \hline 3x^2 - x \\ -(3x^2 + 3x) \\ \hline -4x + 2 \end{array}$$

$$x + 3 + \frac{-4x+2}{x^2+x}$$

Improper Fractions – Algebraic Long Division – Partial Fractions

- Carry out the division to obtain a polynomial and a proper fraction.
- Express the proper fraction in partial fractions.

Improper Fractions – Algebraic Long Division – Partial Fractions

Example:

Express $\dfrac{x^3 - 3x}{x^2 - x - 2}$ in partial fractions

$$\begin{array}{r} x + 1 \\ x^2-x-2 \overline{) x^3 + 0x^2 - 3x + 0} \\ -(x^3 - x^2 - 2x) \\ \hline x^2 - x + 0 \\ -(x^2 - x - 2) \\ \hline 2 \end{array}$$

$$x + 1 + \frac{2}{x^2-x-2}$$

$$\frac{2}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$$\frac{Ax - 2A + Bx + B}{(x+1)(x-2)}$$

$Ax + Bx = 0x$

$A + B = 0$

$-2A + B = 2$

$A + B = 0$

$-3A = 2$

$A = -\frac{2}{3}$

$-\frac{2}{3} + B = 0$

$B = \frac{2}{3}$

$$x + 1 + \frac{-2}{3(x+1)} + \frac{2}{3(x-2)} = \frac{x^3 - 3x}{x^2 - x - 2}$$

(*a*) Express $\dfrac{1}{(x-1)(x+2)^2}$ in partial fractions.

Marks **4**

Express $\dfrac{3x+32}{(x+4)(6-x)}$ in partial fractions and hence evaluate