Proof by Induction

Proving by induction is proving a statement or theorem by a process of ordered steps. It gives us a method of proof for a result which seems to be true.

To prove by induction:

- We first prove that a statement is true for a positive integer value of n e.g. n = 1
- We then check the statement for n = k and assume true.
- We then prove that the theorem is true for n = k + 1 by using the assumption from 2. This will confirm it to be correct.

The conclusion is that if the theorem is true for n = k, then it must be true for n = k + 1 but since it's true for n = 1, then by induction it is true for all positive integers n.

We then pro

Proof by Induction

Examples:

1. For the sequence 1, 3, 6, 10, ... the sequence is generated by the recurrence relation $u_{n+1} = u_n + n + 1$.

The conjecture for the n^{th} term is $u_n = {}^1/_2 n(n+1)$ $n \in N$

Prove this conjecture by induction.

Let n = 1 $\quad u_{1+1} = u_1 + 1 + 1$ $\quad u_1 = \frac{1}{2}(1)(1+1)$

$u_2 = u_1 + 2$ $\quad u_1 = \frac{1}{2}(2) = 1$ True

$3 = 1 + 2$

$3 = 3$ True

Let n = k $\quad u_k = \frac{1}{2}k(k+1)$ Assume true

R.H.S

Let n = k+1 $\quad u_{k+1} = \frac{1}{2}(k+1)(k+2)$

Also $u_{n+1} = u_n + n + 1$

L.H.S

$u_{k+1} = u_k + k + 1$

$u_{k+1} = \frac{1}{2}k(k+1) + k + 1$

$= (k+1)(\frac{1}{2}k + 1)$

$= (k+1)\frac{1}{2}(k+2)$

$= \frac{1}{2}(k+1)(k+2)$ = R.H.S True for n = k+1

Since true for n = k, true for n = k + 1 and true for n = 1 then true for all n≥ 1. n ∈ N.

Proof by Induction

2. Prove that $n^2 + 3n$ is divisible by 2 for all $n \in N$

Let n=1

$\Rightarrow 1^2 + 3(1)$

$= 4$ which is divisible by 2 therefore true.

Let n=k $\quad k^2 + 3k$ assume true

so we let $k^2 + 3k = 2m$

(2m represents a number that is divisible by all multiples of 2).

Let n=k+1 $\quad (k+1)^2 + 3(k+1)$

$k^2 + 2k + 1 + 3k + 3$

$k^2 + 3k + 2k + 4$

$2m + 2k + 4$

$2(m + k + 2)$ which is divisible by 2. Therefore true for n=k+1

The statement is true for n=k, therefore true for n=k+1 and since true for n=1, then true for all n ∈ N.

Proof by Induction

3. Prove by induction that $\sum_{r=1}^{n}(2r-1) = n^2$ for all positive integers n.

Let n = 1 $\quad$ L.H.S $\quad$ R.H.S

$2(1) - 1$ $\quad 1^2$

$= 2 - 1$ $\quad = 1$

$= 1$

L.H.S = R.H.S $\Rightarrow$ True for n = 1

Let n=k $\quad \sum_{r=1}^{k}(2r-1) = k^2$ assume true.

Let n=k+1 $\quad$ L.H.S $\quad$ R.H.S

$\sum_{r=1}^{k+1}(2r-1)$ $\quad (k+1)^2$

$\underbrace{\sum_{r=1}^{k}(2r-1)} + (2(k+1) - 1)$

$k^2 + 2k + 2 - 1$

$k^2 + 2k + 1$

$(k+1)(k+1)$

$= (k+1)^2$ = R.H.S $\Rightarrow$ true for n=k+1

If true for n=k, then true for n=k+1 and since true for n=1 then true for all n ∈ N.

Proof by Induction

4. Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Let n=1 $\quad$ L.H.S $\quad$ R.H.S

$1^2 = 1$ $\quad = \frac{1}{6}(1+1)(2+1)$

$= \frac{1}{6}(2)(3)$

$= 1$

L.H.S = R.H.S True for n = 1

Let n=k $\quad \sum_{r=1}^{k} k^2 = \frac{1}{6}k(k+1)(2k+1)$ Assume true

Let n=k+1 $\quad$ L.H.S $\quad$ R.H.S

$\sum_{r=1}^{k+1}(k+1)^2$ $\quad \frac{1}{6}(k+1)(k+2)(2k+2+1)$

$\frac{1}{6}(k+1)(k+2)(2k+3)$

$\sum_{r=1}^{k} k^2 + (k+1)^2$

$\frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$= k+1\left[\frac{1}{6}k(2k+1) + (k+1)\right]$

$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$

$= \frac{1}{6}(k+1)\left[2k^2 + k + 6k + 6\right]$

$= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$

Proof by Induction

5. Given z = cosθ + isinθ prove by induction that z^n = cosnθ + isinnθ

Let n=1

$z^1 = \cos\theta + i\sin\theta \Rightarrow$ True for n=1

Let n=k

$z^k = \cos k\theta + i\sin k\theta$ assume true

Let n=k+1 $\quad$ L.H.S $\quad$ R.H.S

z^{k+1} $\quad \cos(k+1)\theta + i\sin(k+1)\theta$

$z^k . z^1$

$(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$\cos k\theta\cos\theta + \cos k\theta i\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$\cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$\cos(k+1)\theta + i\sin(k+1)\theta$ = R.H.S True for n = k+1

Since true for n=k, then true for n=k+1 and since true for n=1, therefore true for all n ∈ N.

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1), \quad \forall n \in \mathbb{N}.$$ 4

Let $n=1$ L.H.S $1(3(1)-1) = 1(2) = 2$ R.H.S $1^2(1+1) = 1(2) = 2$ $\Rightarrow$ True for $n=1$

Let $n=k$ $\sum_{r=1}^{k} k(3k-1) = k^2(k+1)$ Assume true

Let $n=k+1$

L.H.S $\sum_{r=1}^{k+1} (k+1)(3k+3-1)$

R.H.S $(k+1)^2(k+1+1)$
$(k+1)^2(k+2)$

$\sum_{r=1}^{k} k(3k-1) + (k+1)(3k+3-1)$

$= k^2(k+1) + (k+1)(3k+2)$

$= k+1(k^2+3k+2)$

$= (k+1)(k+2)(k+1)$

$= (k+1)^2(k+2)$ $=$ R.H.S True for $n=k+1$

If true for n = k, then true for n = k + 1 and since true for n = 1

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} (4r^3 + 3r^2 + r) = n(n+1)^3$$ 6

Let $n=1$ L.H.S $4(1)^3+3(1)^2+1$ $4+3+1$ $=8$ R.H.S $1(1+1)^3$ $=1(2)^3$ $=1(8)=8$ True for $n=1$

Let $n=k$ $\sum_{r=1}^{k} (4k^3+3k^2+k) = k(k+1)^3$ Assume true

Let $n=k+1$

LHS

$\sum_{r=1}^{k+1} (4(k+1)^3+3(k+1)^2+k+1)$

$\sum_{r=1}^{k} (4k^3+3k^2+k) + (4(k+1)^3+3(k+1)^2+k+1)$

$= k(k+1)^3 + 4(k+1)^3+3(k+1)^2+(k+1)$

$= k+1(k(k+1)^2+4(k+1)^2+3(k+1)+1)$

$= k+1(k(k^2+2k+1)+4(k^2+2k+1)+3k+4)$

$= k+1(k^3+2k^2+k+4k^2+8k+4+3k+4)$

$= k+1(k^3+6k^2+12k+8)$ $=$ R.H.S $\Rightarrow$ True for $n=k+1$

R.H.S

$(k+1)(k+1+1)^3$

$=(k+1)(k+2)^3$

$=k+1(k+2)(k^2+4k+4)$

$k+1(k^3+4k^2+4k+2k^2+8k+8)$

$=k+1(k^3+6k^2+12k+8)$

If true for n = k, then true for n = k + 1 and since true for n = 1, then true for all n E N.