Prove by Induction that $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$

Equation of a line in 3 dimensions

2 dimensional lines: We need a coordinate and the gradient.

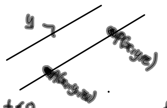
3 dimensional lines: We need a coordinate and a vector in the direction of the line.

Equation of a line in symmetric form

Consider the line that passes through A(x_1, y_1, z_1) parallel to $\underline{u}$ = ai + bj + ck.

(The vector $\underline{u}$ is known as a direction vector for the line L).

Let P(x, y, z) be any point on the line L



$\underline{u} \parallel \overrightarrow{AP}$

Hence

$\overrightarrow{AP} = t\underline{u} \quad t \neq 0$

t is a scalar (2 vectors are parallel when one is a multiple of the other).

$\overrightarrow{AP} = \underline{p} - \underline{a} = \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = t\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ta \\ tb \\ tc \end{pmatrix}$

Equating components

$x - x_1 = ta \qquad y - y_1 = tb \qquad z - z_1 = tc$

$t = \frac{x - x_1}{a} \qquad t = \frac{y - y_1}{b} \qquad t = \frac{z - z_1}{c}$

$$\boxed{t = \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}}$$

The equation of a line can be written in this form. It is known as expressing the equation of line L in symmetric form.

Equation of a line in symmetric form

Example:

(a) The line L passes through the point A(1, -2, 8) and is parallel to the vector 3i + 5j + 11k. Find the equation of the line L in symmetric form.

$x_1 \; y_1 \; z_1$

a b c

(b) Show that the point B(-2, -7, -3) also lies on the line L.

(a) $t = \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

$$\underline{\underline{\frac{x-1}{3} = \frac{y+2}{5} = \frac{z-8}{11}}}$$

(b) Sub. into each equation

$\frac{x-1}{3} = \frac{-2-1}{3} = -1$

$\frac{y+2}{5} = \frac{-7+2}{5} = -1$

$\frac{z-8}{11} = \frac{-3-8}{11} = -1$

Satisfies all three equations, therefore point B lies on the line.

Equation of a line in symmetric form

Find, in symmetric form the equation of the line passing through A(2, -1, 3) and B(3, 2, 5)

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \begin{matrix} a \\ b \\ c \end{matrix}$

$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

$$\underline{\underline{\frac{x-2}{1} = \frac{y+1}{3} = \frac{z-3}{2}}}$$

The equation of a line in parametric form

$\frac{x - x_1}{a} = t \qquad \frac{y - y_1}{b} = t \qquad \frac{z - z_1}{c} = t$

$x - x_1 = at \qquad y - y_1 = bt \qquad z - z_1 = ct$

$x = x_1 + at \qquad y = y_1 + bt \qquad z = z_1 + ct$

Expressing x, y, and z in terms of the parameter t is known as expressing the equation in parametric form. (Any letter can be used instead of t).

The equation of a line in parametric form

Examples:

1. A line passes through $\overset{x_1,\,y_1,\,z_1}{(3, 2, -5)}$ and is parallel to the vector $\overset{a}{2i} + \overset{b}{4j} - \overset{c}{k}$. Find the equation of this line in parametric form.

$$t = \frac{x-3}{2} = \frac{y-2}{4} = \frac{z+5}{-1}$$

$$t = \frac{x-3}{2} \qquad t = \frac{y-2}{4} \qquad t = \frac{z+5}{-1}$$

$$2t = x-3 \qquad 4t = y-2 \qquad -t = z+5$$

$$\underline{\underline{x = 2t+3}} \qquad \underline{\underline{y = 4t+2}} \qquad \underline{\underline{z = -t-5}}$$

Vector form of the equation of a line

To find the vector equation of a line through a point A(x, y, z).

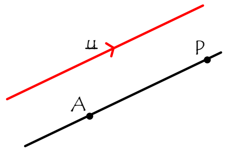
We use the point A and the direction vector $\underline{u}$ where $\underline{u}$ is the direction vector parallel to the line.

$$\overrightarrow{AP} = t\underline{u}$$

$$\underline{p} - \underline{a} = t\underline{u}$$

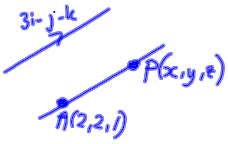
$$\boxed{\underline{p} = \underline{a} + t\underline{u}}$$

Vector equation



Example: Find the eqution of the line passing through A(2, 2, 1) and parallel to 3i – j – k in Vector, Parametric and Symmetric form.

(Cartesian)



Vector form

$$\underline{p} = \underline{a} + t\underline{u}$$

$$\underline{p} = \begin{pmatrix}2\\2\\1\end{pmatrix} + t\begin{pmatrix}3\\-1\\-1\end{pmatrix}$$

or

$$\underline{p} = 2i+2j+k+t(3i-j-k)$$

Parametric form

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix} + \begin{pmatrix}3t\\-t\\-t\end{pmatrix}$$

$$\underline{x = 2+3t,\ y = 2-t,\ z = 1-t}$$

Symmetric or Cartesian form

$$\underline{\frac{x-2}{3} = \frac{y-2}{-1} = \frac{z-1}{-1}}$$

The intersection of two lines

The point of intersection of 2 lines in 3D space can be found when the equations of both lines are expressed in parametric form.

Example:

L_1: x = 8 – 2t, y = –4 + 2t, z = 3 + t

L_2: $\frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2} = s$

Show that these lines intersect and find their point of intersection.

First express L_2 in parametric form

$$L_2:\ x = -2s, \quad \frac{y+2}{-1} = s, \quad \frac{z-9}{2} = s$$

$$y+2 = -s \qquad z-9 = 2s$$

$$y = -s-2 \qquad z = 2s+9$$

Equate coordinates

$$x = 8-2t \qquad y = -4+2t \qquad z = 3+t$$

$$x = -2s \qquad y = -s-2 \qquad z = 2s+9$$

$$8-2t = -2s \qquad -4+2t = -s-2 \qquad 3+t = 2s+9$$

$$① \ -2s+2t = 8 \qquad ② \ 2t+s = 2 \qquad ③ \ 2s-t = -6$$

Solve ① and ②

$$\begin{array}{r} -2s+2t = 8 \\ \ominus \quad s+2t = 2 \\ \hline -3s \quad = 6 \\ \underline{\underline{s = -2}} \end{array} \qquad \begin{array}{l} 2t+(-2) = 2 \\ 2t = 4 \\ \underline{\underline{t = 2}} \end{array}$$

Check these values in ③

$s = -2$ $t = 2$

$$2(-2)-2$$

$$-4-2$$

$$= -6 = \text{RHS}$$

The values of s and t satisfy equation ③, therefore L_1 and L_2 intersect.

Point of Intersection:

t = 2

$$x = 8-2t \qquad y = 2t-4 \qquad z = 3+t$$

$$x = 8-2(2) \qquad y = 2(2)-4 \qquad z = 3+2$$

$$\underline{\underline{x = 4}} \qquad \underline{\underline{y = 0}} \qquad \underline{\underline{z = 5}}$$

$$\underline{\underline{(4, 0, 5)}}$$

Note:

If 2 lines don't intersect, they must be parallel or skew (non-parallel but will never intersect).

The angle between two lines

The angle between 2 intersecting lines L_1 and L_2 is the angle between the direction vectors of the lines and is therefore found as follows:

1. Find a vector $\underline{a}$ in the direction of L_1

2. Find a vector $\underline{b}$ in the direction of L_2

3. The angle between the lines L_1 and L_2 is the angle between the direction vectors $\underline{a}$ and $\underline{b}$.

Hence $\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$

The equation of each line should be expressed in symmetric form first, as the direction vector of a line can be easily read from this form.

The angle between two lines

Examples:

$$L_1 = \frac{x+3}{-1} = \frac{y-4}{1} = \frac{z}{-1}$$

$$L_2 = x = 2t-2, y = -3t+1, z = t-1$$

Given that L_1 and L_2 intersect, calculate the size of the acute angle between the lines.

Write L_2 in symmetric form

$$t = \frac{x+2}{2} = \frac{y-1}{-3} = \frac{z+1}{1}$$

Direction vector for L_1

$$\underline{a} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

Direction vector for L_2

$$\underline{b} = \begin{pmatrix}2\\-3\\1\end{pmatrix}$$

Angle between L_1 and L_2 is angle between $\underline{a}$ and $\underline{b}$

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\underline{a}.\underline{b} = (-1)(2)+(1)(-3)+(-1)(1)$$

$$= -2-3\ 1$$

$$= \underline{\underline{-6}}$$

$$|\underline{a}| = \sqrt{(-1)^2+1^2+(-1)^2} = \sqrt{3}$$

$$|\underline{b}| = \sqrt{2^2+(-3)^2+1^2} = \sqrt{14}$$

$$\cos\theta = \frac{-6}{\sqrt{14}\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{-6}{\sqrt{14}\sqrt{3}}\right) = \underline{157.8}° = \text{Obtuse angle}$$

$$\text{Acute angle} = 180° - 157.8° = \underline{\underline{22.2°}}$$

MARK

14. Two lines L_1 and L_2 are given by the equations:

$$L_1: \quad x = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = -7\lambda$$

$$L_2: \quad \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3}$$

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection. **5**

(b) Calculate the obtuse angle between the lines L_1 and L_2. **4**

$L_1 : \frac{x-4}{3} = \frac{y-2}{4} = \frac{z}{-7}$ $\quad \underline{a} \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$

$L_2 : \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3}$ $\quad \underline{b} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\underline{a}.\underline{b} = (3)(-2) + 4(1) + (-7)(3) = -23$

$|a| = \sqrt{(3)^2 + 4^2 + (-7)^2} = \sqrt{74}$ $\quad |\underline{b}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}$

$\cos\theta = \frac{-23}{\sqrt{74}\sqrt{14}}$

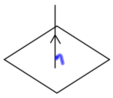
$\theta = \underline{\underline{135.6°}}$

The equation of a plane

A plane is a flat 2D surface.

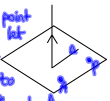
A vector perpendicular to a plane is known as a normal vector and is denoted by $\underset{\sim}{n}$.

A normal vector is in fact perpendicular to all vectors in the plane.



Consider a plane in 3D space relative to a set of coordinate axes.

Let A be a fixed point on the plane and let $\underline{n} = ai + bj + ck$ be a vector normal to the plane passing through A.



Let $P(x, y, z)$ be a typical point on the plane.

$\vec{AP} \perp \underline{n}$

$\underline{n} . \vec{AP} = 0$

$\underline{n} . (\underline{p} - \underline{a}) = 0$

$\underline{n}.\underline{p} - \underline{n}.\underline{a} = 0$

$\underline{n}\underline{p} = \underline{n}.\underline{a}$

$\underline{n}\underline{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + c$

let $\underline{n}.\underline{a} = K$ (k is a constant

∴ Equation of a plane can be express

$\boxed{ax + by + cz = k}$

The coordinates of any point on the plane will

The equation of a plane

Examples:

1. Find the equation of the plane perpendicular to the vector

a b c $\quad$ x y z

i – 3j + 2k and containing P(–1, 2, 1)

$ax + by + cz = k$

$x - 3y + 2z = k$

Sub. P into equation

$-1 - 3(2) + 2(1) = k$

$-5 = k$

The equation of the plane is $\underline{x - 3y + 2z = -5}$

The equation of a plane

x y z

2. Given $\underline{P(2, 1, -4)}$ and Q(1, 2, 3), find the equation of the plane perpendicular to PQ which contains P.

$\vec{PQ} = \underline{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 7 \end{pmatrix}$

Equation of the plane : $-x + y + 7z = k$

$-2 + 1 + 7(-4) = k$

$k = -29$

$\underline{-x + y + 7z = -29}$

The equation of a plane

3. The equation of the line L is given by

$$\frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2}$$

The plane π passes through (1, -4, 2). Find the equation of the plane π.

is perpendicular to the line L and

$\underline{n} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$

x y z

(1, -4, 2)

Equation of the plane

$-2x - y + 2z = k$

$-2(1) - (-4) + 2(2) = k$

$k = 6$

$\underline{-2x - y + 2z = 6}$

x₁ y₁ z₁

A line, L_1, passes through the point P(2, 4, 1) and is parallel to

a b c

$$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$\frac{x-2}{1} = \frac{y-4}{2} = \frac{z-1}{-1}$

and a second line, L_2, passes through Q(–5, 2, 5) and is parallel to

$$\mathbf{u}_2 = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

(a) Write down the vector equations for L_1 and L_2. **2**

(b) Show that the lines L_1 and L_2 intersect and find the point of intersection. **4**

(c) Determine the equation of the plane containing L_1 and L_2. **4**

Vector Product

The scalar product is where two vectors multiply to give a scalar.

The vector product is where 2 vectors can be multiplied to give a vector answer. We write this as $\underline{a} \times \underline{b}$.

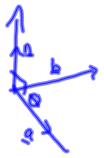
form①

$$\underline{a} \times \underline{b} = \underline{n}|a||b|\sin\theta$$

Where θ is the angle between a and b

$\underline{n}$ is the normal to the plane.

If $\underline{a} = 0$ or $\underline{b} = 0$, then $\underline{n}$ is undefined and $\underline{a} \times \underline{b} = 0$.



Parallel vectors have a vector product of zero.

Vector Product - Component form

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i(a_2b_3 - a_3b_2) - j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)$$

Vector Product

Examples:

1. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= i(2(4) - 3(1)) - j(1(4) - 3(2)) + k(1(1) - 2(2))$$

$$= \underline{5i + 2j - 3k}$$

Vector Product

2. If $\underline{a} = i + 2j + 3k$ and $\underline{b} = 2i - j + k$, find $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = i(2(1) - 3(-1)) - j(1(1) - 3(2)) + k(1(-1) - 2(2))$$

$$= \underline{5i + 5j - 5k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \underline{-5i - 5j + 5k}$$

Vector Product

3. $\underline{a} = 3i + 2j - k$, $\underline{b} = i - j - 2k$ and $c = 4i - 3j + 4k$

find $\underline{a} \times (\underline{b} \times \underline{c})$

$$\underline{b} \times \underline{c} = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ 4 & -3 & 4 \end{vmatrix}$$

$$\underline{b} \times \underline{c} = i(-1(4) - (-2)(-3)) - j(1(4) - (-2)(4)) + k(1(-3) - (-1)(4))$$

$$= \underline{-10i - 12j + k}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -10 & -12 & 1 \end{vmatrix}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = i(2(1) - (-1)(-12)) - j(3(1) - (-1)(-10)) + k(3(-12) - 2(-10))$$

$$= \underline{-10i + 7j - 16k}$$

An important property of the vector product is that the vector $\underline{a} \times \underline{b}$ is always perpendicular to each of the vectors $\underline{a}$ and $\underline{b}$.

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = 0$$
$$\underline{b} \cdot (\underline{a} \times \underline{b}) = 0$$

$$\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$$

Vector Product

An important property of the vector product is that the vector $\underline{a} \times \underline{b}$ is always perpendicular to each of the vectors $\underline{a}$ and $\underline{b}$

Scalar Triple Product

The scalar triple product gives a scalar answer.

Example: Three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ are given by $\underline{u}$, $\underline{v}$ and $\underline{w}$ where $\underline{u}$ = 5i + 13j, $\underline{v}$ = 2i + j + 3k and $\underline{w}$ = i + 4j – k. Calculate $\underline{u}.(\underline{v} \times \underline{w})$

$$\underline{v} \times \underline{w} \quad \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$

$$= i((1)(-1)-(3)(4)) - j(2(-1)-(3)(1)) + k(2(4)-1(1))$$

$$= -13i + 5j + 7k$$

$$u.(\underline{v} \times \underline{w}) = \begin{pmatrix} 5 \\ 13 \\ 0 \end{pmatrix} . \begin{pmatrix} -13 \\ 5 \\ 7 \end{pmatrix} = 5(-13) + 13(5) + (0)(7) = \underline{\underline{0}}$$

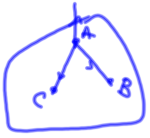
This shows us that $\underline{u}$ is on the same plane as $\underline{v}$ and $\underline{w}$.

Equation of a Plane

We can find the normal to a plane using the vector product. Given a plane containing the vectors $\underline{a}$ and $\underline{b}$, the normal of the plane is $\underline{a} \times \underline{b}$

Examples:

1. Find the equation of the plane containing A(2, 1, 3) B(7, 2, 3) and C(5, 3, 5)



$$\overrightarrow{AB} = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 5 & 1 & 0 \\ 3 & 2 & 2 \end{vmatrix} = i(1(2)-(0)(2)) - j(5(2)-0(3)) + k(5(2)-(1)(3))$$

$$= \overset{a}{2}i - \overset{b}{10}j + \overset{c}{7}k = \underline{n}$$

$$ax + by + cz = k$$

$$2(2) + (-10)(1) + 7(3) = k$$

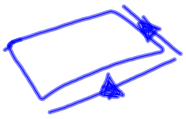
$$k = 15$$

$$\boxed{2x - 10y + 7z = 15}$$

Equation of a Plane

2. A plane is parallel to the vectors $3i + \overset{\underline{u}}{2}j - k$ and $4i \overset{\underline{v}}{-} 2k$. The plane contains the point (1, 1, 0). Find the equation of the plane.

$$\underline{u} \times \underline{v} = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 4 & 0 & -2 \end{vmatrix} = i(2(-2)-(-1)(0)) - j(3(-2)-(-1)(4)) + k(3(0)-4(2))$$



$$= -4i + 2j - 8k = \underline{n}$$

$$ax + by + cz = k$$

$$-4(1) + 2(1) + -8(0) = k$$

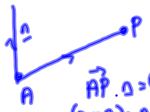
$$k = -2$$

$$\boxed{-4x + 2y - 8z = -2}$$

Equation of a Plane in Vector Form

The Vector Equation of a plane means to leave your equation in the form of vectors.

$$\underline{n}.\underline{p} = \underline{n}.\underline{a}$$



$$\overrightarrow{AP} . \underline{n} = 0$$

$$(\underline{p} - \underline{a}) . \underline{n}$$

Examples:

1. Find the vector equation of the plane through (–1, 2, 1) with normal vector i – 3j + 2k.

$$\underline{n} . \underline{p} = \underline{n} . \underline{a}$$

$$\underline{n} . \underline{p} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} . \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 1(-1) + (-3)(2) + (2)(1) = -5$$

$$\underline{\underline{\underline{p} . (i - 3j + 2k) = -5}}$$

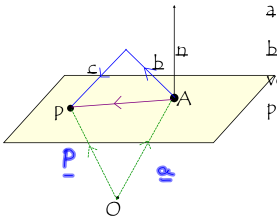
Equation of a Plane in Vector Form

Examples:

2. Find the vector equation of the plane through the point (5, 3, –2) with normal vector 2i – 3j + k.

Parametric Equation of a Plane

We use parallel vectors to work out the parametric equation of a plane.



$\underline{a}$ is the position vector of A.

$\underline{b}$ and $\underline{c}$ are two non parallel vectors that are parallel to the plane.

We describe $\overrightarrow{AP}$ in terms of vectors $\underline{b}$ and $\underline{c}$

$$\overrightarrow{AP} = \lambda \underline{b} + \mu \underline{c}$$

$$\underline{p} - \underline{a} = \lambda \underline{b} + \mu \underline{c}$$

$$\boxed{\underline{p} = \underline{a} + \lambda \underline{b} + \mu \underline{c}}$$

Parametric Equation of a Plane

Examples:

1) Find the parametric equation of the plane which goes through the point $\overset{a}{(5, -2, -3)}$ and parallel to the vectors $\overset{b}{-5i - 2j + 3k}$ and $\overset{c}{-2i + 2j + 4k}$.

$$\underset{\sim}{p} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$$

$$\underset{\sim}{p} = \begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$

$$\underset{\sim}{p} = \underline{\underline{\begin{pmatrix} 5 - 5\lambda - 2\mu \\ -2 - 2\lambda + 2\mu \\ -3 + 3\lambda + 4\mu \end{pmatrix}}}$$

Parametric Equation of a Plane

2) Find the parametric equation of the plane with cartesian equation $x + 5y - 3z = 11$

Parametric Equation of a Plane

3) Find the cartesian form of the equation of a plane with parametric equation

$r = a + \lambda b + \mu c$ where $a = i + 2j - k$

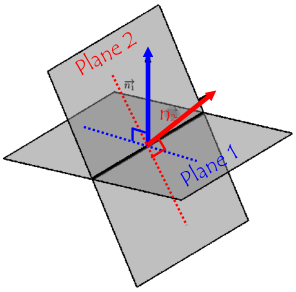
$b = i + j + 2k$

$c = 2j - k$

Equation of a Plane

If two planes are perpendicular, then their normals will be perpendicular. Therefore to show that they are perpendicular, show that $n_1.n_2=0$

The angle between two planes



The angle of intersection of two planes is equal to the angle between the normals of the planes.

$$\text{Cos}\theta = \frac{\underline{n}_1.\underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$$

The angle between two planes

$ax + by + cz = k$

Examples:

π_1 is a plane with equation $2x + 3y + z = 5$ and π_2 is a plane with equation $x + y - z = 0$. Calculate the angle between π_1 and π_2

$\underset{\sim}{n}_1 = 2i + 3j + k$

$\underset{\sim}{n}_2 = i + j - k$

$$\text{Cos}\,\theta = \frac{\underset{\sim}{n}_1 . \underset{\sim}{n}_2}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|}$$

$\underset{\sim}{n}_1 . \underset{\sim}{n}_2 = 2(1) + 3(1) + 1(-1) = 4$

$|\underset{\sim}{n}_1| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

$|\underset{\sim}{n}_2| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$

$$\text{Cos}\,\theta = \frac{4}{\sqrt{14}\sqrt{3}}$$

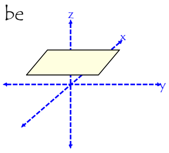
$$\theta = \text{Cos}^{-1}\left(\frac{4}{\sqrt{14}\sqrt{3}}\right)$$

$$\theta = \underline{\underline{51 \cdot 89^\circ}}$$

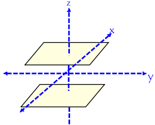
Intersection of planes

In relation to each other, planes in 3D space can be

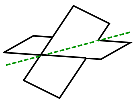
1. Coincident (one on top of the other)



2. Parallel



3. Intersect on a line



Note: parallel or coincident planes have normals that are proportional

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Intersection of 2 planes

If two planes intersect at a line, we determine the equation of the line of intersection.

Example:

The equations of 2 planes are $\overset{\pi_1}{x - 4y + 2z = 1}$ and $\overset{\pi_2}{x - y - z = -5}$. Find the equation of the line of intersection of the two planes.

Let $z = t$

$\pi_1 : x - 4y + 2t = 1$ $\quad$ $\pi_2 : x - y - t = -5$

① $x - 4y = -2t + 1$ $\quad$ ② $x - y = t - 5$

Solve ① and ②

$$\begin{array}{r} x - 4y = -2t + 1 \\ \ominus\; x - y = t - 5 \\ \hline -3y = -3t + 6 \\ \boxed{y = t - 2} \end{array}$$

Sub. into ② $\quad x - (t-2) = t - 5$

$\boxed{x = 2t - 7}$ $\quad$ $\boxed{z = t}$

Parametric equations of Intersection

$x = 2t - 7,\ y = t - 2,\ z = t$

Cartesian form : $\frac{x+7}{2} = \frac{y+2}{1} = \frac{z}{1}$

Intersection of three planes

Three planes can

- Intersect at a line.
- Intersect at a point.
- Not intersect at all.

Intersection of three planes

Examples

1) Intersection at a Point

Find the point of intersection of the 3 planes with the following equations.

$$\begin{aligned} x - 2y + z &= 8 \\ 3x + y - z &= 1 \\ 2x - 2y + 3z &= 18 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 3 & 1 & -1 & 1 \\ 2 & -2 & 3 & 18 \end{array}\right)$$

$$r_2 - 3r_1 \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 7 & -4 & -23 \\ 2 & -2 & 3 & 18 \end{array}\right) r_3 - 2r_1 \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 7 & -4 & -23 \\ 0 & 2 & 1 & 2 \end{array}\right)$$

$$7r_3 - 2r_2 \left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 0 & 7 & -4 & -23 \\ 0 & 0 & 15 & 60 \end{array}\right)$$

$\underline{z = 4}$

$7y - 4(4) = -23$

$\underline{y = -1}$

$x - 2(-1) + 4 = 8$

$\underline{x = 2}$

Pt. of Intersection = $(\underline{2, -1, 4})$

Intersection of three planes

2) Intersection at a line

Find the symmetric equation of the line of intersection of the planes with the following equations:

$$\begin{aligned} x + 2y - 2z &= -7 \\ x - 2y + z &= 6 \\ 3x + 2y - 3z &= -8 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 1 & -2 & 1 & 6 \\ 3 & 2 & -3 & -8 \end{array}\right)$$

$$r_2 - r_1 \left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & -4 & 3 & 13 \\ 3 & 2 & -3 & -8 \end{array}\right) r_3 - 3r_1 \left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & -4 & 3 & 13 \\ 0 & -4 & 3 & 13 \end{array}\right) r_3 - r_2$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & -4 & 3 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$r_1 : x + 2y - 2z = -7$ $\quad$ $r_2 : -4y + 3z = 13$ $\quad$ $\boxed{\text{let } z = t}$

$-4y + 3t = 13$

$-4y = -3t + 13$

$\underline{y = \frac{3}{4}t - \frac{13}{4}}$ $\quad$ $\underline{z = t}$

$x + 2\left(\frac{3}{4}t - \frac{13}{4}\right) - 2t = -7$

$x + \frac{3}{2}t - \frac{13}{2} - 2t = -7$

$x - \frac{1}{2}t = -\frac{1}{2}$

$\underline{x = \frac{1}{2}t - \frac{1}{2}}$

$2x = t - 1$ $\quad$ $y = \frac{3}{4}t - \frac{13}{4}$

$t = 2x + 1$ $\quad$ $4y = 3t - 13$

$t = \frac{4y + 13}{3}$ $\quad$ $t = z$

$t = \underline{\frac{2x+1}{1} = \frac{4y+13}{3} = \frac{z}{1}}$

Intersection of three planes

3) No points of intersection

Gaussian elimination will yield no solution.

Show that these planes do not intersect.

$$\begin{aligned} x + y + z &= 2 \\ 2x - 2y + z &= 5 \\ 3x - y + 2z &= -1 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -2 & 1 & 5 \\ 3 & -1 & 2 & -1 \end{array}\right)$$

$$r_2 - 2r_1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & 1 \\ 3 & -1 & 2 & -1 \end{array}\right) r_3 - 3r_1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & 1 \\ 0 & -4 & -1 & -7 \end{array}\right)$$

$$r_3 - r_2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & -1 & 1 \\ 0 & 0 & 0 & -8 \end{array}\right)$$

Inconsistent set of equations

$\Rightarrow$ no solution $\Rightarrow$ planes do not intersect.

The Intersection of a line and a plane

If there is only one solution then a point of intersection exists.

If there are many solutions then the line lies on the plane.

If there is no solution then the line is parallel to the plane.

(To show that a line is parallel to the plane you need to show

that the direction vector of the line is perpendicular to normal vector of the plane. ie d.n = 0)

The Intersection of a line and a plane

1. Find the equation of the plane in parametric form.
2. Substitute parametric form with x, y and z into the equation of the plane.
3. Solve the equation for the parameter and hence find the coordinate of the point of intersection.

The Intersection of a line and a plane

Example:

Find the point of intersection of the line

$$t = \frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$$

with the plane with equation 2x + y - z = 4

$\frac{x-3}{4} = t$ $\quad$ $\frac{y-2}{-1} = t$ $\quad$ $\frac{z+1}{2} = t$

$\underline{x = 4t+3}$ $\quad$ $\underline{y = -t+2}$ $\quad$ $\underline{z = 2t-1}$

$2(4t+3) + (-t+2) - (2t-1) = 4$

$8t + 6 - t + 2 - 2t + 1 = 4$

$5t = -5$

$\underline{\underline{t = -1}}$

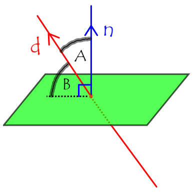
$x = 4(-1)+3$ $\quad$ $y = -(-1)+2$ $\quad$ $z = 2(-1)-1$

$\underline{x = -1}$ $\quad$ $\underline{y = 3}$ $\quad$ $\underline{z = -3}$

$(-1, 3, -3)$ = pt. of Intersection.

The angle between a line and a plane

- Find the normal vector to the plane
- Find the direction vector from the line
- Use the scalar product to find the angle between 'n' and 'd'.
- Remember to subtract the angle from 90° to find the angle between the line and the plane. (B = 90° - A)



The angle between a line and a plane

Example:

Calculate the size of the acute angle between the line

$$\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$$

and the plane with equation 2x + y - z = 4

$\underline{a} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ $\quad$ $\underline{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $\quad$ $\cos\theta = \frac{\underline{a}.\underline{n}}{|\underline{a}||\underline{n}|}$

$\underline{a} . \underline{n} = (4)(2) + (-1)(1) + 2(-1) = 5$

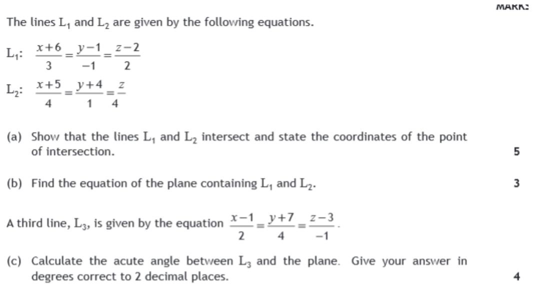
$|\underline{a}| = \sqrt{4^2 + (-1)^2 + 2^2} = \sqrt{21}$

$|\underline{n}| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

$\cos\theta = \frac{5}{\sqrt{21}\sqrt{6}}$ $\quad$ $\theta = \cos^{-1}\left(\frac{5}{\sqrt{21}\sqrt{6}}\right) = 63.5°$

$90° - 63.5° = \underline{\underline{26.5°}}$

Specimen



MARK

The lines L_1 and L_2 are given by the following equations.

L_1: $\frac{x+6}{3} = \frac{y-1}{-1} = \frac{z-2}{2}$

L_2: $\frac{x+5}{4} = \frac{y+4}{1} = \frac{z}{4}$

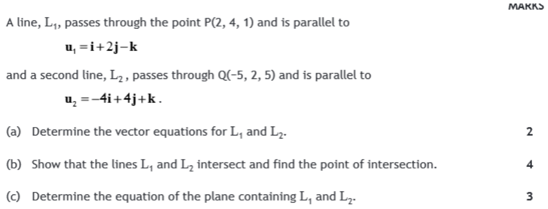
(a) Show that the lines L_1 and L_2 intersect and state the coordinates of the point of intersection. 5

(b) Find the equation of the plane containing L_1 and L_2. 3

A third line, L_3, is given by the equation $\frac{x-1}{2} = \frac{y+7}{4} = \frac{z-3}{-1}$.

(c) Calculate the acute angle between L_3 and the plane. Give your answer in degrees correct to 2 decimal places. 4

Exemplar


MARKS

A line, L$_1$, passes through the point P(2, 4, 1) and is parallel to

$\mathbf{u}_1 = \mathbf{i}+2\mathbf{j}-\mathbf{k}$

and a second line, L$_2$, passes through Q(−5, 2, 5) and is parallel to

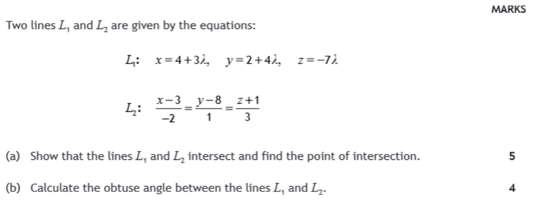
$\mathbf{u}_2 = -4\mathbf{i}+4\mathbf{j}+\mathbf{k}$.

(a) Determine the vector equations for L$_1$ and L$_2$. 2

(b) Show that the lines L$_1$ and L$_2$ intersect and find the point of intersection. 4

(c) Determine the equation of the plane containing L$_1$ and L$_2$. 3

2016


MARKS

Two lines L_1 and L_2 are given by the equations:

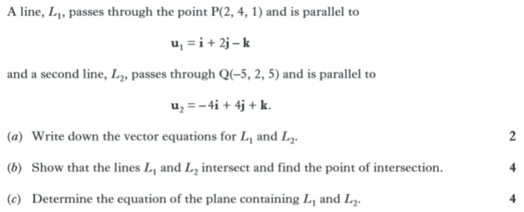
L_1: $x = 4+3\lambda,\quad y = 2+4\lambda,\quad z = -7\lambda$

L_2: $\frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3}$

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection. 5

(b) Calculate the obtuse angle between the lines L_1 and L_2. 4

2015


A line, L_1, passes through the point P(2, 4, 1) and is parallel to

$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

and a second line, L_2, passes through Q(−5, 2, 5) and is parallel to

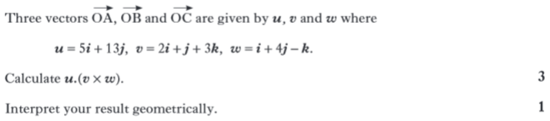
$\mathbf{u}_2 = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

(*a*) Write down the vector equations for L_1 and L_2. 2

(*b*) Show that the lines L_1 and L_2 intersect and find the point of intersection. 4

(*c*) Determine the equation of the plane containing L_1 and L_2. 4

2014

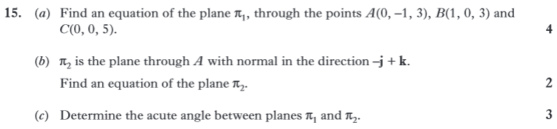

Three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are given by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ where

$\boldsymbol{u} = 5\boldsymbol{i} + 13\boldsymbol{j},\ \boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k},\ \boldsymbol{w} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$.

Calculate $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})$. 3

Interpret your result geometrically. 1

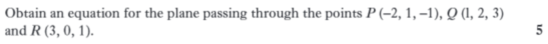
2013


15. (*a*) Find an equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$ and $C(0, 0, 5)$. 4

(*b*) π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$.
Find an equation of the plane π_2. 2

(*c*) Determine the acute angle between planes π_1 and π_2. 3

2012


Obtain an equation for the plane passing through the points P (−2, 1, −1), Q (1, 2, 3) and R (3, 0, 1). 5