Complex Numbers

Complex Numbers were first introduced in the 16th century by an Italian mathematician called Cardano. He referred to them as ficticious numbers.

Given an equation that does not give real roots such as

$$x^2 + 3x + 25 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{-91}}{2}$$

let $\sqrt{-1} = i$

$$x = \frac{-3 \pm \sqrt{91}\sqrt{-1}}{2}$$

$$x = \frac{-3 \pm \sqrt{91}\, i}{2}$$

Complex Numbers

$$\boxed{\sqrt{-1} = i}$$

By writing $\sqrt{-1}$ as i, we can now solve equations that were previously seen as impossible.

This also means that $\boxed{i^2 = -1}$

Can you think about expressions for i^3 and i^4?

$i^3 = \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1}$

$-1 \times \sqrt{-1} = -i$

$i^4 = \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1}$

$= 1$

What can you then say about i^{2n} and i^{2n+1} where n is an integer?

Complex Numbers

Complex Numbers became more acceptable to use in the 18th century.

They are known as complex because they are made up of real and imaginary parts. The complex number z is represented below in its Cartesian form.

$$z = a + ib$$

Real part known as Re(z)

Imaginary part known as Im(z)

Complex Numbers +/−/x

We can add, subtract, multiply and divide complex numbers. They follow the same type of rules as algebra.

Examples: Given $z_1 = 4 + 3i$ and $z_2 = 1 - 2i$, find the values of

1. $z_1 + z_2$ $= 4 + 3i + 1 - 2i$
   $= 5 + i$
2. $z_1 - z_2$ $= 4 + 3i - 1 + 2i$
   $= 3 + 5i$
3. $z_1 z_2$ $= (4+3i)(1-2i)$
   $4 - 8i + 3i - 6i^2$
   $4 - 5i + 6$ $\boxed{i^2 = -1}$
   $10 - 5i$
4. z_1^2 $= (4+3i)^2$
   $16 + 24i + 9i^2$ $\boxed{i^2 = -1}$
   $16 + 24i - 9$
   $7 + 24i$

Complex Numbers

Two complex numbers are equal if and only if the real and imaginary parts of the complex numbers are equal.

$$\boxed{a + bi = c + di \iff a = c \text{ and } b = d}$$

Example: Given that $x + 2yi = 3 + (x + 1)i$ where $x, y \in \mathbb{R}$, find x and y

$x + 2yi = 3 + xi + i$

Equate like terms

$x = 3$

$2yi = (x+1)i$

$2yi = 4i$

$y = 2$

Solving a quadratic equation to give complex roots

Example: Solve $z^2 - 4z + 13 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36}\sqrt{-1}}{2}$$

$\boxed{\sqrt{-1} = i}$

$$x = \frac{4 \pm 6i}{2}$$

$$x = 2 \pm 3i$$

Roots: $2 + 3i$, $2 - 3i$

Division of Complex Numbers

Remember rationalising the denominator for surds?

$$\frac{3}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{3+3\sqrt{2}}{-1} = -3-3\sqrt{2}$$

$(1-\sqrt{2})(1+\sqrt{2})$
$1+\sqrt{2}-\sqrt{2}-\sqrt{4}$
$1-2=-1$

Note that the complex roots of a quadratic function are a complex number and its conjugate. E.g $5 \pm 3i$

The product of a complex number and its conjugate gives a real number. This is helpful for division of complex numbers.

Division of Complex Numbers

z represents a complex number & $\bar{z}$ represents its conjugate.

Examples: Express the following in the form x + iy where x, y $\in$ R

1. $\frac{8+i}{3+2i} \times \frac{(3-2i)}{(3-2i)} = \frac{(8+i)(3-2i)}{(3+2i)(3-2i)} = \frac{24-16i+3i-2i^2}{9-6i+6i-4i^2}$

$\boxed{i^2=-1}$ $\frac{24-13i+2}{9+4} = \frac{26-13i}{13} = 2-i$

2. $\frac{1-7i}{4-3i} \times \frac{4+3i}{4+3i} = \frac{4+3i-28i-21i^2}{16+12i-12i-9i^2}$ $\boxed{i^2=-1}$

$= \frac{25-25i}{25} = 1-i$

Complex Numbers & Square Roots

Example: Find two solutions for which $z = \sqrt{5-12i}$

Let $z = a+ib$

$a+ib = \sqrt{5-12i}$

$(a+ib)^2 = 5-12i$

$a^2+2aib+i^2b^2 = 5-12i$

$a^2-b^2+2aib = 5-12i$

Equate like terms

$a^2-b^2=5$ $\quad 2aib=-12i$

$ab=-6$

$a=\frac{-6}{b}$

$\left(\frac{-6}{b}\right)^2-b^2=5$

$\frac{36}{b^2}-b^2=5$

$36-b^4=5b^2$

$b^4+5b^2-36=0$

$(b^2-4)(b^2+9)=0$

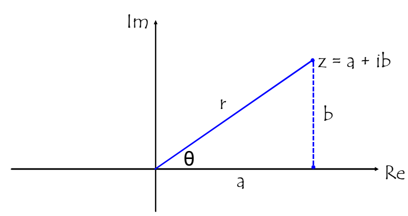
$b^2=4$ or $b^2\neq-9$ (b is real)

$b=\pm2$

finish

Argand Diagrams & Polar Form

Argand diagrams are a way of geometrically representing complex numbers. We plot them on a Complex Plane where x represents the real axis and the y - axis represents imaginary numbers (except (0, 0)).



The line representing z is similar to a vector because it has magnitude and direction.

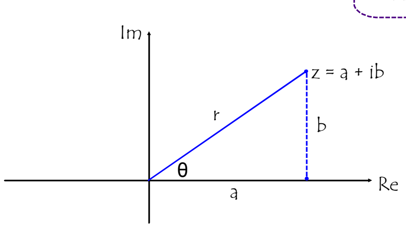
Argand Diagrams & Polar Form

The modulus of z is the distance from the origin to z and is known as |z| or r.

We can calculate it by using Pythagoras

$a^2 + b^2 = r^2$

$r = |z| = \sqrt{a^2+b^2}$



Argand Diagrams & Polar Form

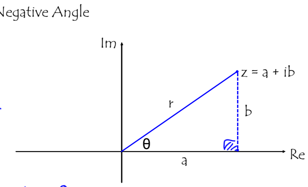
- θ is the angle in radians between Oz and the positive direction of the x - axis and is known as the argument of z or Arg z.
- The Principal Value of the argument is the value which lies between $-\pi$ (180°) and π (−180°) and is written arg z. Anti-Clockwise => Positive Angle, Clockwise => Negative Angle

$\cos\theta = \frac{a}{r}$

$a = r\cos\theta$

$\sin\theta = \frac{b}{r}$

$b = r\sin\theta$



$z = a+ib$

$z = r\cos\theta + ir\sin\theta$

We can write a + ib in what's known as Polar Form using trigonometry.

We can express a and b using r and θ

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

r = magnitude $\quad$ θ = arg z

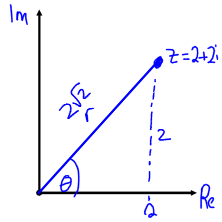
Argand Diagrams & Polar Form

Always start by plotting the complex number on an Argand Diagram.

Examples: Find the modulus and the argument of the following and hence express in Polar Form

1. $z = 2 + 2i$

$r(\cos\theta + i\sin\theta)$



$r = \sqrt{a^2+b^2}$

$r = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$\arg z = \theta$

$\sin\theta = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

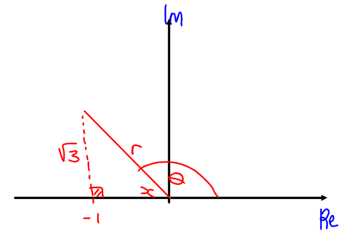
$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Argand Diagrams & Polar Form

Examples: Find the modulus and the argument of the following and hence express in Polar Form

2. $z = -1 + \sqrt{3}i$



$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\arg z$

$\tan x = \frac{\sqrt{3}}{1} = \sqrt{3}$

$x = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

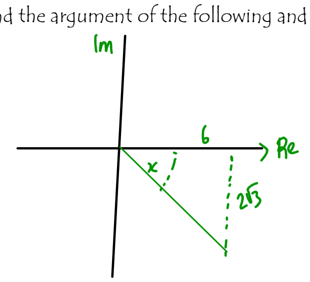
$z = r(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Argand Diagrams & Polar Form

Examples: Find the modulus and the argument of the following and hence express in Polar Form

3. $z = 6 - 2\sqrt{3}i$



$r = \sqrt{6^2 + (-2\sqrt{3})^2}$

$= \sqrt{48} = 4\sqrt{3}$

$\arg z \Rightarrow \sin x = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2}$

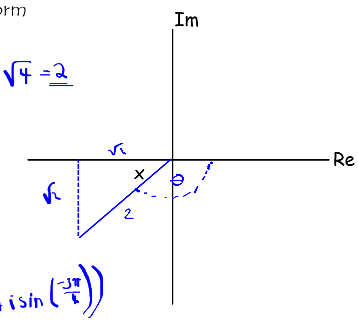
$x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$z = 4\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

Argand Diagrams & Polar Form

Examples: Find the modulus and the argument of the following and hence express in Polar Form

4. $z = -\sqrt{2} - \sqrt{2}i$



$r = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{4} = 2$

$\arg z \Rightarrow \tan x = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$x = \tan^{-1}(1) = \frac{\pi}{4}$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$z = 2\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

Polar form to Cartesian Form

Examples:

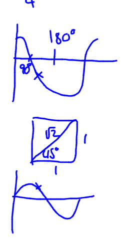
1. Express $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ in Cartesian form using exact values.

$\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$-\frac{\sqrt{2}}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}}i$

$-1 + i$

$\frac{3\pi}{4} = 135°$



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3. Express the complex number $z = -i + \frac{1}{1-i}$ in the form $z = x + iy$ stating the values of x and y. 3 marks

Find the modulus and argument of z and plot z and $\bar{z}$ on an Argand diagram. 4 marks

Polar Form: Multiplication and Division

Given two complex numbers, we can use Addition Formulae from Higher to get a simplified expression for the product and quotient.

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1\cos\theta_2 + \cos\theta_1 \, i\sin\theta_2 + \cos\theta_2 \, i\sin\theta_1 + i^2 \sin\theta_1 \sin\theta_2]$

$r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)]$

$r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$\left.\begin{array}{l}\text{* Multiply the modulii} \\ \text{* Add the arguments.}\end{array}\right\}$ Multiplying in Polar Form

Dividing in Polar Form:

$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

* Divide the modulii
* Subtract the arguments.

Polar Form: Multiplication and Division

Examples:

1. Calculate $z_1 z_2$ when $z_1 = 3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

and $z_2 = 5(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

$z_1 z_2 = 15\left(\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)\right)$

$= 15(\cos\pi + i\sin\pi)$

In Cartesian Form $= 15(-1+0i) = \underline{-15}$

Polar Form: Multiplication and Division

Examples:

2. Given $z = 8(\cos 50^\circ + i\sin 50^\circ)$ and $w = 2(\cos 30^\circ + i\sin 30^\circ)$

Find (a) zw (b) $\frac{z}{w}$ (c) $\frac{w^3}{z^2}$

and give your answers in the form $r(\cos\theta + i\sin\theta)$

(a) $zw = 16\underline{(\cos 80^\circ + i\sin 80^\circ)}$

(b) $\frac{z}{w} = 4\underline{(\cos 20^\circ + i\sin 20^\circ)}$

(c) $w^3 = 8(\cos 90^\circ + i\sin 90^\circ)$ $z^2 = 64(\cos 100^\circ + i\sin 100^\circ)$

$\frac{w^3}{z^2} = \underline{\frac{1}{8}(\cos(-10^\circ) + i\sin(-10)^\circ)}$

De Moivre's Theorem

Given $z = r(\cos\theta + i\sin\theta)$, we notice that

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$

Therefore $\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$

De Moivre's Theorem

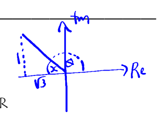
To raise a complex number to a power:

- change the complex number into polar form
- use De Moivre's Theorem
- change back into cartesian form (if required)

Note: If you are leaving it in Polar Form and the range exceeds ±180°, just +/- 360° to it.

De Moivre's Theorem

Examples:



1. $z = -\sqrt{3} + i$

(a) Express z^4 in the form $x + iy$, $x, y \in \mathbb{R}$

(b) Show that $z^6 + 64 = 0$

(a) $r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$

$\tan x = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{6}$ $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$z = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$

$z^4 = 2^4\left(\cos\left(\frac{5\pi}{6}\times 4\right) + i\sin\left(\frac{5\pi}{6}\times 4\right)\right)$

$= 16(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3})$

$16\left(-\frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)i\right)$

$\underline{-8 - 8\sqrt{3}i}$

(b) $z^6 = 2^6\left(\cos\left(\frac{5\pi}{6}\times 6\right) + i\sin\left(\frac{5\pi}{6}\times 6\right)\right)$

$64(\cos 5\pi + i\sin 5\pi)$

$64(-1)$

$= -64$

$z^6 + 64 = -64 + 64 = \underline{0}$

De Moivre's Theorem

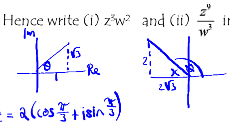
Examples:

2. $z = 1 + \sqrt{3}i$ and $w = -2\sqrt{3} + 2i$

(a) Write z and w in Polar Form

(b) Hence write (i) z^3w^2 and (ii) $\frac{z^9}{w^3}$ in the form $x + iy$ where $x,y \in R$

(a)



$\underline{z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$

$r = \sqrt{(2)^2 + (2\sqrt{3})^2}$
$= \sqrt{16} = 4$

$\tan x = \frac{2}{2\sqrt{3}}$

$x = \frac{\pi}{6}$ $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\underline{w = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

(b) (i) z^3w^2

$z^3 = 8(\cos\pi + i\sin\pi)$
$w^2 = 16\left(\cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}\right)$

$z^3w^2 = 128\left(\cos\frac{16\pi}{6} + i\sin\frac{16\pi}{6}\right)$
$= 128\left(-\frac{1}{2} + i\sin\frac{\sqrt{3}}{2}\right)$
$= \underline{-64 + 64\sqrt{3}i}$

(ii) $\frac{z^9}{w^3}$

$z^9 = 512(\cos 3\pi + i\sin 3\pi)$
$w^3 = 64\left(\cos\frac{15\pi}{6} + i\sin\frac{15\pi}{6}\right)$

$\frac{z^9}{w^3} = 8\left(\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}\right)$
$= 8(0 + 1i)$
$= \underline{8i}$

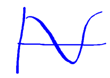
De Moivre's Theorem & Trig. Identities

We can use De Moivre's Theorem to express Complex Numbers with fractional and negative indices too.

Example: $z^n = \cos n\theta + i\sin n\theta$

Given $z = \cos\theta + i\sin\theta$

Write an expression for $\frac{1}{z^n}$ and hence $z^n + \frac{1}{z^n}$



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$\frac{1}{z^n} = z^{-n} = \cos -n\theta + i\sin -n\theta$
$= \underline{\cos n\theta - i\sin n\theta}$

$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$
$= \underline{2\cos n\theta}$

De Moivre's Theorem & Trig. Identities

We can use De Moivre's Theorem & Binomial Theorem to prove some trig. identities.

Reminder from Higher:

$\boxed{\sin 2A = 2\sin A\cos A}$

$\boxed{\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A}$

$\boxed{\sin^2 A + \cos^2 A = 1}$

Note: Powers of i are cyclical

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$i^6 = -1$

De Moivre's Theorem & Trig. Identities

Example: Express $\sin 5\theta$ in terms of $\sin\theta$

$\cos 5\theta + \underline{i\sin 5\theta} = (\cos\theta + i\sin\theta)^5$

Expand R.H.S using Binomial

$\binom{5}{0}(\cos\theta)^5 + \binom{5}{1}(\cos\theta)^4(i\sin\theta) + \binom{5}{2}(\cos\theta)^3(i\sin\theta)^2$
$+ \binom{5}{3}(\cos\theta)^2(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)(i\sin\theta)^4 + \binom{5}{5}(i\sin\theta)^5$

$\cos^5\theta + \underline{5\cos^4\theta\, i\sin\theta} - 10\cos^3\theta\sin^2\theta - \underline{10\cos^2\theta\, i\sin^3\theta}$
$+ 5\cos\theta\sin^4\theta + \underline{i\sin^5\theta}$

Equate (Re) with (Re) and (Im) with (Im)

Im: $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$
$= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$
$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10(\sin^3\theta - \sin^5\theta) + \sin^5\theta$
$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$\sin 5\theta = \underline{-20\sin^3\theta + 5\sin\theta + 16\sin^5\theta}$

De Moivre's Theorem & Trig. Identities

Using the previous example, try now to also express $\cos 5\theta$ in terms of $\cos\theta$

Solving Equations to give Complex Roots

We have already looked at how to solve quadratic equations that give complex roots.

We can also find the roots of equations with higher order polynomials.

The roots occur in conjugate pairs if the polynomial has real coefficients.

If you have been given 1 root, you will be able to find the others.

Solving Equations to give Complex Roots

If the roots of a quadratic equation are $z = -1$ and $z = 3$, then the equation must be $y = (z + 1)(z - 3)$ therefore $y = z^2 - 2z - 3$

Quadratic equations can be formed in this way from the roots (even complex roots).

$$z^2 - z(\text{sum of roots}) + \text{product of roots} = 0$$

Solving Equations to give Complex Roots

Example 1:

Show that $-2 + 2i$ is a root of the equation $z^3 + 3z^2 + 4z - 8 = 0$ and find the other roots

$$(-2+2i)^3 + 3(-2+2i)^2 + 4(-2+2i) - 8$$
$$(-2+2i)(4-8i+4i^2) + 3(4-8i+4i^2) - 8 + 8i - 8$$
$$-8 + 16i - 8i^2 + 8i - 16i^2 + 8i^3 + 12 - 24i + 12i^2 - 8 + 8i - 8$$
$$-8 + 16i + 8 + 8i + 16 - 8i + 12 - 24i - 12 - 8 + 8i - 8$$
$$= 0i + 0 = \underline{\underline{0}} \Rightarrow \underline{-2+2i \text{ is a root}}$$

Conjugate root $= -2 - 2i$

Sum of roots $= -2 + \cancel{2i} + (-2) - \cancel{2i} = \underline{\underline{-4}}$

product $= (-2-2i)(-2+2i) = 4 - \cancel{4i} + \cancel{4i} - 4i^2 = \underline{\underline{8}}$

$$\underline{z^2 + 4z + 8}$$

$$\begin{array}{r} z - 1 \\ z^2+4z+8 \overline{\big)\, z^3 + 3z^2 + 4z - 8} \\ \ominus\ z^3 + 4z^2 + 8z \downarrow \\ \hline -z^2 - 4z - 8 \\ \ominus\ -z^2 - 4z - 8 \\ \hline 0 \end{array}$$

Roots are $z = \underline{\underline{-2+2i}}$, $z = \underline{\underline{-2-2i}}$, $\underline{\underline{z = 1}}$

Solving Equations to give Complex Roots

Example 2:

Verify that $z = 2$ is a roots of the equation $z^3 - 4z^2 + 9z - 10 = 0$. Hence find all the roots of this equation.

$$\begin{array}{r} z^2 - 2z + 5 \\ z-2 \overline{\big)\, z^3 - 4z^2 + 9z - 10} \\ \ominus\ z^3 - 2z^2 \downarrow \\ \hline -2z^2 + 9z \\ \ominus\ -2z^2 + 4z \downarrow \\ \hline 5z - 10 \\ \ominus\ 5z - 10 \\ \hline \underline{\underline{0}} \end{array}$$

→ Remainder $= 0$ $\Rightarrow z = 2$ is a root

$$z^2 - 2z + 5$$

$a = 1$ $b = -2$ $c = 5$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{4 - 20}}{2} \quad (\sqrt{-1} = i)$$
$$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Roots: $z = 2$, $z = 1 + 2i$, $z = 1 - 2i$

Finding nth roots of a complex number

We will now use De Moivre's Theorem to find roots in a different way.

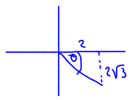
An equation $z^2 = a + ib$ will have 2 roots, $z^3 = a + ib$ will have 3 roots etc. Therefore an equation with z^n will have n roots or solutions.

- We first write the number in Polar Form $r(\cos\theta + i\sin\theta)$
- We can then write our complex number in the form $r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$ for $n = 1$, $n = 2$ etc. without changing its value.
- We use De Moivre's Theorem to solve.

Finding nth roots of a complex number

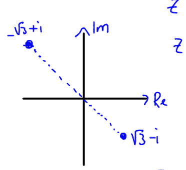
Examples:

1. Solve $z^2 = 2 - 2\sqrt{3}i$ and sketch your solutions on an Argand Diagram



$$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{16} = \underline{\underline{4}}$$

$$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3} \qquad \theta = \frac{-\pi}{3}$$



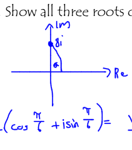
$$z^2 = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$
$$z = \left[4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^{\frac{1}{2}}$$
$$= 2\left(\cos\left(-\frac{\pi}{3} \times \frac{1}{2}\right) + i\sin\left(-\frac{\pi}{3} \times \frac{1}{2}\right)\right)$$
$$= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$
$$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \underline{\underline{\sqrt{3} - i}}$$

$$z = \left[4\left(\cos\left(-\frac{\pi}{3} + 2\pi\right) + i\sin\left(-\frac{\pi}{3} + 2\pi\right)\right)\right]^{\frac{1}{2}}$$
$$= 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)^{\frac{1}{2}}$$
$$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= \underline{\underline{-\sqrt{3} + i}}$$

Finding nth roots of a complex number

Examples:

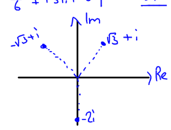
2. Find the 3 cube roots of the complex number $8i$, expressing each root in the form $a + ib$, where $a, b \in R$. Show all three roots on a single Argand Diagram



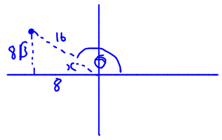
$r = 8$ $\quad \theta = \frac{\pi}{2}$

$$z^3 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$
$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{2} \times \frac{1}{3}\right) + i\sin\left(\frac{\pi}{2} \times \frac{1}{3}\right)\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \underline{\sqrt{3} + i}$$
$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{2} + 2\pi\right) + i\sin\left(\frac{\pi}{2} + 2\pi\right)\right)^{\frac{1}{3}} = 2\left(\cos\left(\frac{5\pi}{2} \times \frac{1}{3}\right) + i\sin\left(\frac{5\pi}{2} \times \frac{1}{3}\right)\right)$$
$$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \underline{-\sqrt{3} + i}$$

$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{2} + 4\pi\right) + i\sin\left(\frac{\pi}{2} + 4\pi\right)\right)^{\frac{1}{3}} = 2\left(\cos\left(\frac{9\pi}{2} \times \frac{1}{3}\right) + i\sin\left(\frac{9\pi}{2} \times \frac{1}{3}\right)\right)$$
$$= 2\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) = \underline{-2i}$$



(d) $z^4 = -8 + 8\sqrt{3}i$



$r = \sqrt{8^2 + (8\sqrt{3})^2} = 16$

$\tan x = \sqrt{3}$

$x = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \underline{\underline{\frac{2\pi}{3}}}$

$z^4 = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

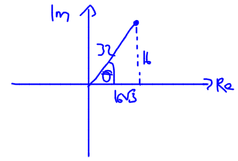
$z_1 = 16^{\frac{1}{4}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \underline{\underline{\sqrt{3} + 1i}}$

$z_2 = 16^{\frac{1}{4}}\left(\cos\frac{8\pi}{12} + i\sin\frac{8\pi}{12}\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \underline{\underline{-1 + \sqrt{3}i}}$

$z_3 = 16^{\frac{1}{4}}\left(\cos\frac{14\pi}{12} + i\sin\frac{14\pi}{12}\right) = 2\left(-\frac{\sqrt{3}}{2} + -\frac{1}{2}i\right) = \underline{\underline{-\sqrt{3} - i}}$

$z_4 = 2\left(\cos\frac{20\pi}{12} + i\sin\frac{20\pi}{12}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \underline{\underline{1 - \sqrt{3}i}}$

(f) $z^5 = 16\sqrt{3} + 16i$



$r = \sqrt{16^2 + (16\sqrt{3})^2} = 32$

$\tan\theta = \frac{16}{16\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$z^5 = 32\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z_1 = 2\left(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\right)$

$z_2 = 2\left(\cos\frac{13\pi}{30} + i\sin\frac{13\pi}{30}\right)$

$z_3 = 2\left(\cos\frac{25\pi}{30} + i\sin\frac{25\pi}{30}\right)$

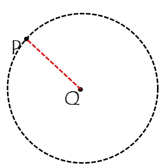
$z_4 = 2\left(\cos\frac{37\pi}{30} + i\sin\frac{37\pi}{30}\right)$

$z_5 = 2\left(\cos\frac{49\pi}{30} + i\sin\frac{49\pi}{30}\right)$

Geometrical Interpretation of Equations on a Complex Plane

The path formed by a point which moves according to some rule is known as its locus.

E.g. The path of the point P given that it always has to be 5cm away from Q i.e. |PQ| = 5. Below is a diagram of the locus of P.



Locus in the Complex Plane

Suppose a complex number z moves in the complex plane subject to some constraint (for example Modulus = 3 or arg(z) = 30°)

The path of the complex number z is known as the locus of z. The equation of the locus can be found (because the locus can make the shape of a circle or line etc).

Locus in the Complex Plane

The modulus of z $|z| = |x + yi| = \sqrt{x^2 + y^2}$

This formula can be used to find the equation of the locus. Recall from higher that the circle with centre (a,b) and radius r is $(x-a)^2 + (y-b)^2 = r^2$

Locus in the Complex Plane

Examples:

1. Given z = x + iy, draw the locus of the point which moves on the complex plane such that:

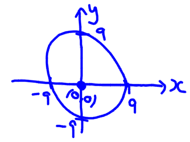
(i) $|z| = 9$ (ii) $|z| \leq 9$ (iii) $|z| \geq 9$

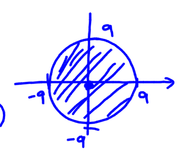
$|z| = \sqrt{x^2 + y^2} = 9$

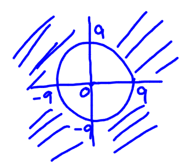
$x^2 + y^2 = 81$

Circle with centre (0,0)

radius = 9







Locus in the Complex Plane

Examples:

2. The complex number z moves in the complex plane subject to the condition |z + 1 - 2i| = 4. Find the equation of the locus and interpret the locus geometrically

Circle with centre $(-1, 2)$
radius $= 4$

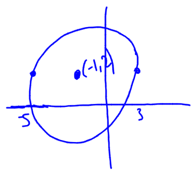
Let $z = x + iy$

$|x + iy + 1 - 2i| = 4$

$|x + 1 + i(y-2)| = 4$

$\sqrt{(x+1)^2 + (y-2)^2} = 4$

$(x+1)^2 + (y-2)^2 = 16$



Locus in the Complex Plane

Examples:

3. The complex number z moves in the complex plane such that |z + 2i| = |z + 3|. Show that the locus of z is a straight line and find its equation.

Let $z = x + iy$

$|x + iy + 2i| = |x + iy + 3|$

$|x + i(y+2)| = |x + 3 + iy|$

$\sqrt{x^2 + (y+2)^2} = \sqrt{(x+3)^2 + y^2}$

$x^2 + y^2 + 4y + 4 = x^2 + 6x + 9 + y^2$

$4y + 4 = 6x + 9$

$4y = 6x + 5$

Straight line equation $\Rightarrow$ locus is straight line.

5 Locus in the Complex Plane

Examples:

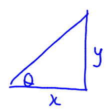
4. Find the equation of the locus and interpret geometrically

|z + i| < 2

Locus in the Complex Plane

Examples:

5. If z = x + iy, find the equation of the locus when arg (z) = $\frac{3\pi}{4}$



$\tan \frac{y}{x} = \frac{3\pi}{4}$

$\tan\left(\frac{3\pi}{4}\right) = -1$

$\frac{y}{x} = -1$ $\boxed{y = -x}$