Solving Systems of Equations

As seen previously, we can solve systems of equations using matrices.

This time we will have more unknowns.

Given a system of equations, we can create an augmented matrix (like we did for 3x3 inverse) to help us find the unknowns. This is called **Gaussian Elimination**.

For example:

$$4x - 3y + z = 5$$
$$2x + y + 2z = 7$$
$$x + 4y + 2z = 8$$

$$\left(\begin{array}{ccc|c} 4 & -3 & 1 & 5 \\ 2 & 1 & 2 & 7 \\ 1 & 4 & 2 & 8 \end{array}\right)$$

Solving Systems of Equations

We perform row operations in order to convert the matrix into upper triangular form (getting all the entries below the main diagonal to be zero).

Examples:

1. Solve the following system of equations for x, y , z.

$$2x - y + z = 5$$
$$x - 3y + 2z = 2$$
$$2x + y + 4z = -3$$

$$2r_2 - r_3 \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & -3 & 2 & 2 \\ 2 & 1 & 4 & -3 \end{array}\right) \xrightarrow[r_3 - r_1]{} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & -7 & 0 & 7 \\ 2 & 1 & 4 & -3 \end{array}\right)$$

$$\xrightarrow{\frac{2}{7}r_2} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & -7 & 0 & 7 \\ 0 & 2 & 3 & -8 \end{array}\right) \xrightarrow[r_3 + r_2]{} \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & -2 & 0 & 2 \\ 0 & 2 & 3 & -8 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 3 & -6 \end{array}\right)$$

$$2x - y + z = 5$$
$$-2y + 0z = 2$$
$$3z = -6$$

$\underline{\underline{z = -2}}$ $\underline{\underline{y = -1}}$

$$2x - (-1) + (-2) = 5$$
$$2x + 1 - 2 = 5$$
$$2x = 6$$
$$\underline{\underline{x = 3}}$$

Solving Systems of Equations

Examples:

2. Solve

$$-2x - 3y + z = 1$$
$$4x - y + z = 17$$
$$x + 3y - z = -4$$

Solving Systems of Equations

If a matrix is continually reduced such that the matrix reduces to the identity, the answers are immediate. This type of reduced matrix is called the matrix in reduced echelon form.

$$x + 2y = 10$$
$$2x - 3y + z = 3$$
$$x - 4z = 18$$

Ans. $x = 6$, $y = 2$, $z = -3$

Solving Systems of Equations

Examples:

3. The points (1, 6) (2, 9) and (-3, 34) lie on a parabola with equation $y = ax^2 + bx + c$ where a, b, and c are constants.

(a) Write down 3 equations in a, b, c

(b) Hence find the equation of the parabola

(a) $6 = a(1)^2 + b(1) + c$ $\quad 9 = a(2)^2 + b(2) + c$ $\quad 34 = a(-3)^2 + b(-3) + c$

$a + b + c = 6$ $\quad 4a + 2b + c = 9$ $\quad 9a - 3b + c = 34$

(b)

$$r_2 - 4r_1 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 9 \\ 9 & -3 & 1 & 34 \end{array}\right) \xrightarrow[r_3 - 9r_1]{} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -15 \\ 9 & -3 & 1 & 34 \end{array}\right)$$

$$r_3 - 6r_2 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -15 \\ 0 & -12 & -8 & -20 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -15 \\ 0 & 0 & +10 & 70 \end{array}\right)$$

$$10c = 70$$
$$\underline{c = 7}$$

$$-2b - 3c = -15$$
$$-2b - 21 = -15$$
$$-2b = 6$$
$$\underline{b = -3}$$

$$a - 3 + 7 = 6$$
$$\underline{a = 2}$$

$$\boxed{2x^2 - 3x + 7 = f(x)}$$

Identifying types of solutions which exist

A system of equations with a solution can be represented as two lines intersecting at a point, hence a unique solution.

Some equations have no solutions and some have infinite solutions.

Parallel lines will give no solution

E.g. $2x - y = 3$

$3y - 6x = -5$

$$r_3 + 3r_1 \left(\begin{array}{cc|c} 2 & -1 & 3 \\ -6 & 3 & -5 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 0 & -5 \end{array}\right)$$

$$0y \neq -5$$

This type of equation is known as inconsistent.

Infinite Solutions

Example:

x + y + z = 1

2x + 2y + z = 3

2x + 2y + 2z = 2

$$\begin{array}{c} \\ r_2 - 2r_1 \\ \\ \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 3 \\ 2 & 2 & 2 & 2 \end{array}\right) \xrightarrow[r_3 - 2r_1]{} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 2 & 2 & 2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad -1z = 1, \; z = -1 \qquad x + y + z = 1, \; x + y - 1 = 1, \; x + y = 2$$

There are an infinite number of possibilities for x and y therefore there are infinite solutions.

The general rule is that if there is a zero on the leading diagonal after reduction to upper triangular form then there is no unique solution to the system of equations.

Infinite Solutions

Ill Conditioning

A system of equations is ill conditioned when a small change in the right hand side of the equation causes a large change in the solution.

E.g. x + 0.99y = 1.99

0.99x + 0.98y = 1.97

$$\begin{array}{c} \\ r_2 - 0.99r_1 \end{array}\left(\begin{array}{cc|c} 1 & 0.99 & 1.99 \\ 0.99 & 0.98 & 1.97 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0.99 & 1.99 \\ 0 & -0.0001 & -0.0001 \end{array}\right)$$

$$-0.0001y = -0.0001, \quad \underline{y = 1}, \quad \underline{x = 1}$$

Ill Conditioning

Consider what happens with a small change to the original equations

so that x + 0.99y = 2.00

0.99x + 0.98y = 1.97

$$\begin{array}{c} \\ r_2 - 0.99r_1 \end{array}\left(\begin{array}{cc|c} 1 & 0.99 & 2 \\ 0.99 & 0.98 & 1.97 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0.99 & 2 \\ 0 & -0.0001 & -0.01 \end{array}\right)$$

$$-0.0001y = -0.01, \quad \underline{\underline{y = 100}} \qquad x + 99 = 2.00, \quad \underline{\underline{x = -97}}$$

Ill Conditioning

Geometrically this occurs when the lines given by 2 equations meet at a very small angle. A small change to the right hand side of 1 of the equations replaces one line with a parallel line very close by but gives an intersection that is considerably further away.