Implicit Differentiation

In functions of the form y = f(x), y is expressed in terms of x. We say that the variable y is given explicitly in terms of x.

E.g. $y = 3x^2 - 4x + 9$

In a function where y and x are intermixed and not of the form y = f(x), we say that these functions are implicit. They imply that a function exists.

E.g. $x^2 + 2xy - 2y = 4$

An implicit function can be expressed explicitly but when it is too difficult to do this, we differentiate using implicit differentiation.

Implicit Differentiation

Note: When we are differentiating $3x^2$, we say $\frac{d}{dx}3x^2$ (when the function isn't $y = 3x^2$)

When y is a term in the function, we differentiate using the chain rule.

E.g. y^4 $\quad 4y^3 \cdot {}^{dy}/_{dx}$

Implicit Differentiation

Examples: Find dy/dx for each of these functions

1. $x^4 = x^2 - y^2$

$4x^3 = 2x - 2y \cdot {}^{dy}/_{dx}$

$2y \, {}^{dy}/_{dx} = 2x - 4x^3$

${}^{dy}/_{dx} = \frac{2x - 4x^3}{2y} = \frac{x(1-2x^2)}{y}$

Implicit Differentiation

Examples: Find dy/dx for each of these functions

2. $x^2 - 3xy + y^2 = 4$

$2x - 3x({}^{dy}/_{dx}) + y(-3) + 2y \cdot {}^{dy}/_{dx} = 0$

$2y \, {}^{dy}/_{dx} - 3x({}^{dy}/_{dx}) = 3y - 2x$

${}^{dy}/_{dx}(2y - 3x) = 3y - 2x$

${}^{dy}/_{dx} = \frac{3y - 2x}{2y - 3x}$

Implicit Differentiation

Examples: Find dy/dx for each of these functions

3. $3xy + 5y^2 - x^2 = 2y$

$3x({}^{dy}/_{dx}) + y(3) + 10y \, {}^{dy}/_{dx} - 2x = 2 \, {}^{dy}/_{dx}$

$3x({}^{dy}/_{dx}) + 10y \, {}^{dy}/_{dx} - 2 \, {}^{dy}/_{dx} = 2x - 3y$

${}^{dy}/_{dx}(3x + 10y - 2) = 2x - 3y$

${}^{dy}/_{dx} = \frac{2x - 3y}{3x + 10y - 2}$

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Implicit Differentiation

Examples: Find dy/dx for each of these functions

4. $\ln(x + y) = \tan^{-1} x$ Inverse trig. fucntions example

Equations of Tangents

Example:

A curve is defined by the implicit equation $x^2 + y^2 + 2x - 4y = 15$

Find the equation of the tangent at the point (3, 4) on the curve.

$m = dy/dx$

$d/dx(x^2+y^2+2x-4y=15)$

$= 2x + 2y\,dy/dx + 2 - 4\,dy/dx = 0$

$dy/dx(2y-4) = -2x-2$

$dy/dx = \dfrac{-2x-2}{2y-4} = \dfrac{-x-1}{y-2}$ at (3,4)

$m = \dfrac{-3-1}{4-2} = \dfrac{-4}{2} = -2$

$y - 4 = -2(x-3)$

$\underline{y = -2x+10}$

Implicit Differentiation and the second derivative

Examples: Find dy/dx and d^2y/dx^2

1. $x^2 + 2xy = 1$

$2x + 2x\left(dy/dx\right) + y(2) = 0$

$2x\,dy/dx = -2x - 2y$

$dy/dx = \dfrac{-2x-2y}{2x} = \underline{\dfrac{-x-y}{x}}$

$d^2y/dx^2 = \dfrac{x(-1-dy/dx) - (-x-y)(1)}{x^2}$

$= \dfrac{x(-1-dy/dx)+x+y}{x^2}$

$= \dfrac{x\left(-1-\left(\frac{-x-y}{x}\right)\right)+x+y}{x^2}$

$= \dfrac{\cancel{x}\left(\frac{-x+x+y}{\cancel{x}}\right)+x+y}{x^2}$

$d^2y/dx^2 = \dfrac{y+x+y}{x^2} = \underline{\dfrac{2y+x}{x^2}}$

Starter 7.9.2017

The equation $x^4 + y^4 + 9x - 6y = 14$ defines a curve passing through the point A(1, 2).

Obtain the equation of the tangent to the curve at A. **4**

$4x^3 + 4y^3\,dy/dx + 9 - 6\,dy/dx = 0$

$dy/dx\,(4y^3-6) = -4x^3-9$

$dy/dx = \dfrac{-4x^3-9}{4y^3-6}$ at A(1,2)

$m = \dfrac{-4(1)^3-9}{4(2)^3-6} = \dfrac{-13}{26} = \underline{\dfrac{-1}{2}}$

$y - 2 = -\tfrac{1}{2}(x-1)$

$\underline{2y = -x+5}$

Implicit Differentiation and the second derivative

Examples: Find dy/dx and d^2y/dx^2

2. $xy^2 + 2 = y$

$x(2y\,dy/dx) + y^2(1) = dy/dx$

$\Rightarrow 2xy\,dy/dx + y^2 = dy/dx$

$\Rightarrow dy/dx(2xy-1) = -y^2$

$\Rightarrow dy/dx = \underline{\dfrac{-y^2}{2xy-1}}$ or $\underline{\dfrac{y^2}{1-2xy}}$

$d^2y/dx^2 = \dfrac{(1-2xy)(2y\,dy/dx) - y^2(-2x(dy/dx)+y(-2))}{(1-2xy)^2}$

$= \dfrac{2y\,dy/dx - 4xy^2\,dy/dx + 2xy^2\,dy/dx + 2y^3}{(1-2xy)^2}$

$= \dfrac{dy/dx\,(2y - 4xy^2 + 2xy^2) + 2y^3}{(1-2xy)^2}$

$= \dfrac{\left(\frac{y^2}{1-2xy}\right)(2y-2xy^2) + 2y^3}{(1-2xy)^2}$

$= \dfrac{\frac{2y^3}{1-2xy} - \frac{2xy^4}{1-2xy} + 2y^3}{(1-2xy)^2}$

$= \dfrac{2y^3 - 2xy^4 + 2y^3(1-2xy)}{(1-2xy)^3}$

$= \dfrac{2y^3 - 2xy^4 + 2y^3 - 4xy^4}{(1-2xy)^3}$

$= 4y^3 - 6xy^4 \quad 2y^3(2y-3xy)$

Q4. From HW sheet

If $y = \dfrac{\sin x}{x^2}$, prove that $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + (x^2+2)y = 0$

$dy/dx = \dfrac{x^2(\cos x) - \sin x.(2x)}{x^4} = \dfrac{x^2\cos x - 2x\sin x}{x^4} = \underline{\dfrac{x\cos x - 2\sin x}{x^3}}$

$d^2y/dx^2 \Rightarrow \dfrac{x^3\left[x(-\sin x)+\cos x.(1) - 2(\cos x) + \sin x(0)\right] - 3x^2\left[x\cos x - 2\sin x\right]}{x^6}$

$\Rightarrow \dfrac{x^3\left[-x\sin x + \cos x - 2\cos x\right] - 3x^3\cos x + 6x^2\sin x}{x^6}$

$\Rightarrow \dfrac{-x^4\sin x + x^3\cos x - 2x^3\cos x - 3x^3\cos x + 6x^2\sin x}{x^6}$

$\Rightarrow \dfrac{\cancel{x^2}\left[-x^2\sin x + x\cos x - 2x\cos x - 3x\cos x + 6\sin x\right]}{x^{\cancel{6}\,4}}$

$d^2y/dx^2 = \dfrac{-4x\cos x + (6-x^2)\sin x}{x^4}$

L.H.S

$\cancel{x^2}\left(\dfrac{-4x\cos x + (6-x^2)\sin x}{\cancel{x^4}\,x^2}\right) + 4\cancel{x}\left(\dfrac{x\cos x - 2\sin x}{\cancel{x^3}\,x^2}\right) + (x^2+2)\left(\dfrac{\sin x}{x^2}\right)$

$\dfrac{-\cancel{4x\cos x} + \cancel{6\sin x} - \cancel{x^2\sin x} + \cancel{4x\cos x} - \cancel{8\sin x} + \cancel{x^2\sin x} + \cancel{2\sin x}}{x^2}$

$= \dfrac{0}{x^2} = 0 = $ R.H.S

$\underline{\text{QED}}$

② (c) $\sin^2\left(2x - \frac{\pi}{6}\right)$

$\left[\sin\left(2x-\frac{\pi}{6}\right)\right]^2$

$2\left[\sin\left(2x-\frac{\pi}{6}\right)\right].\,2\cos\left(2x-\frac{\pi}{6}\right)$

$= \underline{4\sin\left(2x-\frac{\pi}{6}\right)\cos\left(2x-\frac{\pi}{6}\right)}$

(g) $\frac{x^2 \ln x}{x+1}$

$f'(x) \Rightarrow \frac{x+1\left(x^2\left(\frac{1}{x}\right)+\ln x \cdot (2x)\right)-x^2 \ln x \,(1)}{(x+1)^2}$

$\Rightarrow \frac{x+1\left(x+2x\ln x\right)-x^2\ln x}{(x+1)^2}$

$\Rightarrow \frac{x^2+2x^2\ln x+x+2x\ln x-x^2\ln x}{(x+1)^2}$

$\Rightarrow \frac{x^2+x+2x\ln x+x^2\ln x}{(x+1)^2}$

$\Rightarrow \frac{x(x+1+2\ln x+x\ln x)}{(x+1)^2}$

$y = x^3e^{-x}$

$dy/dx = -x^3e^{-x} + 3x^2e^{-x} = 0$ at s.p.'s

$x^2e^{-x}(-x+3) = 0$

$x^2e^{-x} = 0$ $\quad \underline{x=3}$

$\underline{x=0}$ $\quad e^{-x} \neq 0$ no solution

$y = x^3e^{-x}$
$y = 27e^{-3}$
$y = 0$

$d^2y/dx^2 = x^3e^{-x} - 6x^2e^{-x} + 6xe^{-x}$ rising

at $x=0$ $\quad d^2y/dx^2 = 0$ $\quad (0,0)$ pt. of Inflection

at $x=3$ $\quad d^2y/dx^2 = -0.44\ldots$ Max. at $(3, 27e^{-3})$