Integration by parts

We use integration by parts when we have a product and one of the factors is not related to the other (we can't use substitution).

E.g. $\int x \sin x dx$

The formula is derived from the product rule

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrate both sides

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$= \boxed{\int u\,dv = uv - \int v\,du}$$

Pick a 'u' that is easy to differentiate and a 'dv' that is integratable.

Integration by parts

Examples:

$$\int u\,dv = uv - \int v\,du$$

1. $\int x \cos 2x dx$

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$dv = \cos 2x\,dx$

$\frac{dv}{dx} = \cos 2x$

$v = \int \cos 2x\,dx$

$\boxed{v = \frac{1}{2}\sin 2x}$

$$\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$$

$$\frac{1}{2}x\sin 2x - \left[-\frac{1}{4}\cos 2x + C\right]$$

$$\underline{\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C}$$

Integration by parts

Examples:

2. $\int x\sin 3x dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$u = x$

$\frac{du}{dx} = 1$

$\boxed{du = 1dx}$

$dv = \sin 3x dx$

$\frac{dv}{dx} = \sin 3x$

$\boxed{v = -\frac{1}{3}\cos 3x}$

$-\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x dx$

$\underline{-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x dx + C}$

Integration by parts

Examples:

3. $\int xe^{4x}dx$

$$\int u\,dv = uv - \int v\,du$$

let $u = x$ and $dv = e^{4x}dx$

$\frac{du}{dx} = 1$

$\boxed{du = 1dx}$

$\frac{dv}{dx} = e^{4x}$

$v = \frac{e^{4x}}{4} + C$

$x\left(\frac{e^{4x}}{4} + c\right) - \int \frac{e^{4x}}{4} + C\,dx$

$\frac{1}{4}xe^{4x} + Cx - \left[\frac{1}{16}e^{4x} + Cx + C\right]$

$\frac{1}{4}xe^{4x} + \cancel{Cx} - \frac{1}{16}e^{4x} - \cancel{Cx} - C$

$\underline{\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C}$

Integration by parts

Examples:

$$\int u\,dv = uv - \int v\,du$$

4. $\int (2x+1)e^{-2x}dx$

let $u = 2x+1$ and $dv = e^{-2x}dx$

$\frac{du}{dx} = 2$ $\quad$ $\frac{dv}{dx} = e^{-2x}$

$\boxed{du = 2dx}$ $\quad$ $v = -\frac{1}{2}e^{-2x}$

$$= (2x+1)\left(-\tfrac{1}{2}e^{-2x}\right) - \int 2\left(-\tfrac{1}{2}e^{-2x}\right)dx$$

$$-xe^{-2x} - \tfrac{1}{2}e^{-2x} - \left[\tfrac{1}{2}e^{-2x}\right] + C$$

$$-xe^{-2x} - e^{-2x} + C$$

$$\underline{e^{-2x}\left[-x-1\right] + C}$$

Integration by parts

Examples:

$$\int u\,dv = uv - \int v\,du$$

5. $\int x^3 \ln x\,dx$

$u = \ln x$ and $dv = x^3 dx$

$\frac{du}{dx} = \frac{1}{x}$ $\quad$ $\frac{dv}{dx} = x^3$ $\quad$ $v = \frac{x^4}{4}$

$du = \frac{1}{x}dx$

$$\frac{x^4}{4}\ln x - \int \frac{x^4}{4}\left(\frac{1}{x}\right)dx$$

$$\frac{x^4}{4}\ln x - \int \frac{x^3}{4}dx$$

$$\underline{\frac{x^4}{4}\ln x - \frac{x^4}{16} + C}$$

Integration by parts

Examples:

$$\int u\,dv = uv - \int v\,du$$

6. $\int \frac{1}{x^4}\ln x\,dx$

let $u = \ln x$ and $dv = x^{-4}dx$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow \boxed{du = \frac{1}{x}dx}$ $\quad \frac{dv}{dx} = x^{-4}$

$v = -\frac{1}{3}x^{-3}$

$-\frac{1}{3}x^{-3}\ln x - \int \underbrace{\left(-\frac{1}{3}x^{-3}\right)\left(\frac{1}{x}\right)}dx \rightarrow -\frac{1}{3x^3} \times \frac{1}{x} = -\frac{1}{3x^4} = -\frac{1}{3}x^{-4}$

$= -\frac{1}{3x^3}\ln x - \int -\frac{1}{3}x^{-4}\,dx$

$= -\frac{1}{3x^3}\ln x - \left[\frac{1}{9}x^{-3}\right] + C$

$= -\frac{1}{3x^3}\ln x - \frac{1}{9x^3} + C$

Integration by parts

Examples:

$$\int u\,dv = uv - \int v\,du$$

7. $\int x\sqrt{2x-1}\,dx$

let $u = x$ and $dv = (2x-1)^{\frac{1}{2}}dx$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = (2x-1)^{\frac{1}{2}}$

$du = 1dx$

$v = \frac{(2x-1)^{3/2}}{(3/2)(2)} = \frac{(2x-1)^{3/2}}{3}$

$\frac{x(2x-1)^{\frac{3}{2}}}{3} - \int \frac{1}{3}(2x-1)^{3/2}dx$

$\frac{1}{3}x(2x-1)^{\frac{3}{2}} - \frac{1}{3}\left[\frac{(2x-1)^{5/2}}{\frac{5}{2}(2)}\right] + C$

$\frac{1}{3}x(2x-1)^{3/2} - \frac{(2x-1)^{5/2}}{15} + C$

Integration by parts

Examples:

$$\int u\,dv = uv - \int v\,du$$

8. $\int \sin^{-1} x dx$

let $u = \sin^{-1}x$ $\quad dv = 1dx$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{dv}{dx} = 1$

$du = \frac{1}{\sqrt{1-x^2}}dx$ $\quad v = x$

$x\sin^{-1}x - \underbrace{\int \frac{x}{\sqrt{1-x^2}}dx}$

let $u = 1-x^2$

$\frac{du}{dx} = -2x$

$du = -2xdx$

$-\frac{1}{2}du = xdx$

$-\frac{1}{2}\int u^{-\frac{1}{2}}du$

$-\frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]$

$= -u^{\frac{1}{2}}$

$= -(1-x^2)^{\frac{1}{2}}$

$\underline{x\sin^{-1}x + \sqrt{1-x^2} + C}$

Integration by parts - Repeated Application

Somtimes we have to use integration by parts more than once.

Examples:

$$\int u\,dv = uv - \int v\,du$$

1. $\int x^2 \cos 3x dx$

let $u = x^2$ $\quad dv = \cos 3x dx$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = \cos 3x$

$du = 2xdx$ $\quad v = \frac{1}{3}\sin 3x$

$\frac{1}{3}x^2\sin 3x - \int \frac{1}{3}(2x)\sin 3x\, dx$

$\frac{1}{3}x^2\sin 3x - \frac{2}{3}\underbrace{\int x\sin 3x dx}_{\text{Integration by parts again!}}$

$\boxed{u=x}$ $\quad dv = \sin 3x dx$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = \sin 3x$

$\boxed{du = 1dx}$ $\quad \boxed{v = -\frac{1}{3}\cos 3x}$

$-\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x dx$

$-\frac{1}{3}x\cos 3x + \left[\frac{1}{3}\left(\frac{1}{3}\right)\sin 3x\right]$

$-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x$

$\frac{1}{3}x^2\sin 3x - \frac{2}{3}\left[-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right]$

$\underline{\frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x}$

Integration by parts - Repeated Application

2. $\int x^2 e^x dx$

$u = x^2 \quad dv = e^x dx$

$\frac{du}{dx} = 2x \quad \frac{dv}{dx} = e^x$

$\boxed{du = 2x\,dx} \quad \boxed{v = e^x}$

$x^2e^x - 2\underbrace{\int xe^x dx}$

$u = x \quad dv = e^x dx$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = e^x$

$\boxed{du = 1dx} \quad \boxed{v = e^x}$

$xe^x - \int e^x dx$

$xe^x - e^x$

$x^2e^x - 2[xe^x - e^x]$

$x^2e^x - 2xe^x + 2e^x + C$

Integration by parts - Definite Integrals

$\int u\,dv = uv - \int v\,du$

1. $\int_0^{\frac{\pi}{2}} x \sin 2x dx$

$u = x \quad dv = \sin 2x\,dx$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \sin 2x$

$v = -\frac{1}{2}\cos 2x$

$du = 1dx$

$\left[-\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\,dx\right]_0^{\frac{\pi}{2}}$

$= \left[-\frac{1}{2}x\cos 2x + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \left[-\frac{\pi}{4}\cos(\pi) + \frac{1}{4}\sin\pi\right] - \left[0 + \frac{1}{4}\sin 0\right]$

$= \frac{\pi}{4}$

2. $\int_0^1 xe^{-2x} dx$

$u = x \quad dv = e^{-2x}dx$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = e^{-2x}$

$\boxed{du = 1dx} \quad v = \frac{e^{-2x}}{-2}$

$\boxed{v = -\frac{1}{2}e^{-2x}}$

$\left[-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}dx\right]_0^1$

$= \left[-\frac{1}{2}xe^{-2x} + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]\right]_0^1$

$= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1$

$= \left[-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right] - \left[-\frac{1}{4}\right]$

$= -\frac{3}{4}e^{-2} + \frac{1}{4}$

$= \frac{1}{4}(1 - 3e^{-2})$

Integration by parts - PP Questions

Specimen Paper

Find $\int x^2 e^{3x} dx$. **5**

Integration by parts - PP Questions

Obtain $\int x^7 (\ln x)^2 \, dx$. **6**

Integration by parts - PP Questions

Obtain the exact value of $\int_0^2 x^2 e^{4x}\,dx$. **5**