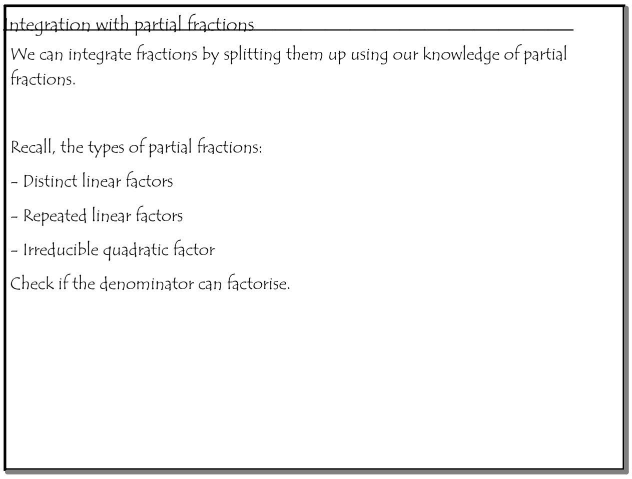


Integration with partial fractions

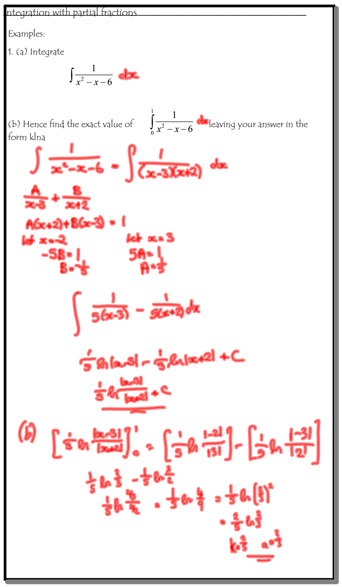
We can integrate fractions by splitting them up using our knowledge of partial fractions.

Recall, the types of partial fractions:

- Distinct linear factors
- Repeated linear factors
- Irreducible quadratic factor

Check if the denominator can factorise.

Dec 6-07:58



ntegration with partial fractions

Examples:

1. (a) Integrate

$$\int \frac{1}{x^2-x-6}\,dx$$

(b) Hence find the exact value of $\int_0^1 \frac{1}{x^2-x-6}\,dx$ leaving your answer in the form klna

$$\int \frac{1}{x^2-x-6} = \int \frac{1}{(x-3)(x+2)}\,dx$$

$$\frac{A}{x-3}+\frac{B}{x+2}$$

$A(x+2)+B(x-3)=1$

let $x=-2$ let $x=3$

$-5B=1$ $5A=1$

$B=-\frac{1}{5}$ $A=\frac{1}{5}$

$$\int \frac{1}{5(x-3)} - \frac{1}{5(x+2)}\,dx$$

$$\frac{1}{5}\ln|x-3| - \frac{1}{5}\ln|x+2| + C$$

$$\frac{1}{5}\ln\frac{|x-3|}{|x+2|}+C$$

(b) $\left[\frac{1}{5}\ln\frac{|x-3|}{|x+2|}\right]_0^1 = \left[\frac{1}{5}\ln\frac{|-2|}{|3|}\right] - \left[\frac{1}{5}\ln\frac{|-3|}{|2|}\right]$

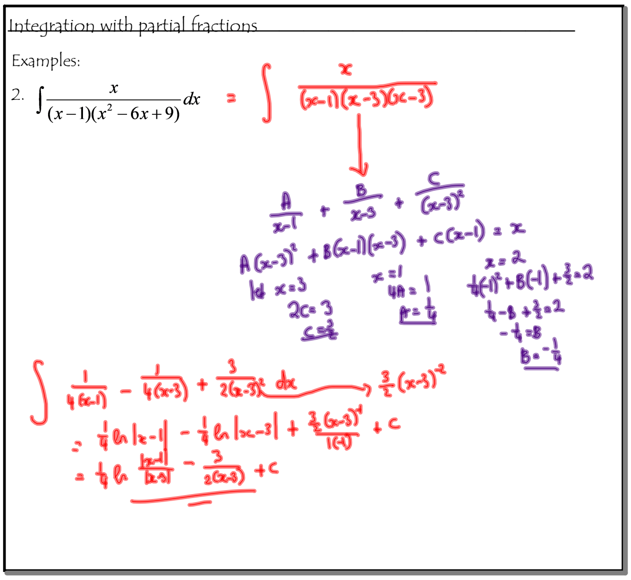
$\frac{1}{5}\ln\frac{2}{3} - \frac{1}{5}\ln\frac{3}{2}$

$\frac{1}{5}\ln\frac{2/3}{3/2} = \frac{1}{5}\ln\frac{4}{9} = \frac{1}{5}\ln\left(\frac{2}{3}\right)^2$

$= \frac{2}{5}\ln\frac{2}{3}$

$k=\frac{2}{5}$ $a=\frac{2}{3}$

Dec 6-09:56



Integration with partial fractions

Examples:

2. $\int \frac{x}{(x-1)(x^2-6x+9)}dx = \int \frac{x}{(x-1)(x-3)(x-3)}$

$$\frac{A}{x-1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$$

$A(x-3)^2+B(x-1)(x-3)+C(x-1)=x$

let $x=3$ $x=1$ $x=2$

$2C=3$ $4A=1$ $\frac{1}{4}(-1)^2+B(-1)+\frac{3}{2}=2$

$C=\frac{3}{2}$ $A=\frac{1}{4}$ $\frac{1}{4}-B+\frac{3}{2}=2$

$-\frac{1}{4}=B$

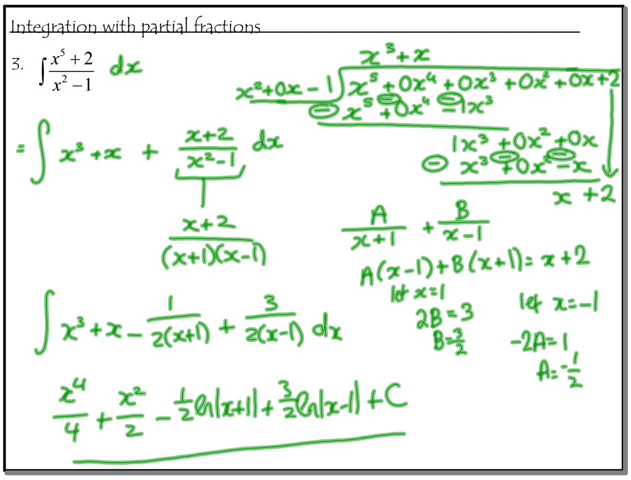
$B=-\frac{1}{4}$

$$\int \frac{1}{4(x-1)} - \frac{1}{4(x-3)} + \frac{3}{2(x-3)^2}\,dx \longrightarrow \frac{3}{2}(x-3)^{-2}$$

$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x-3| + \frac{\frac{3}{2}(x-3)^{-1}}{1(-1)} + c$

$= \frac{1}{4}\ln\frac{|x-1|}{|x-3|} - \frac{3}{2(x-3)} + c$

Dec 6-14:49



Integration with partial fractions

3. $\int \frac{x^5+2}{x^2-1}\,dx$

$$\begin{array}{r} x^3+x \\ x^2+0x-1\,\big)\,\overline{x^5+0x^4+0x^3+0x^2+0x+2} \\ \ominus\; x^5+0x^4-1x^3 \\ \hline 1x^3+0x^2+0x \\ \ominus\; x^3+0x^2-x \\ \hline x+2 \end{array}$$

$= \int x^3+x + \frac{x+2}{x^2-1}\,dx$

$$\frac{x+2}{(x+1)(x-1)}$$

$$\frac{A}{x+1}+\frac{B}{x-1}$$

$A(x-1)+B(x+1)=x+2$

let $x=1$ let $x=-1$

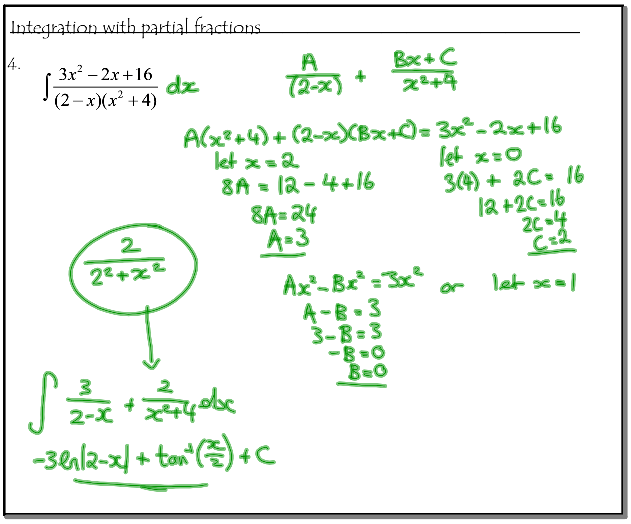
$2B=3$ $-2A=1$

$B=\frac{3}{2}$ $A=-\frac{1}{2}$

$$\int x^3+x-\frac{1}{2(x+1)}+\frac{3}{2(x-1)}\,dx$$

$$\frac{x^4}{4}+\frac{x^2}{2}-\frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|+C$$

Dec 6-14:51



Integration with partial fractions

4. $\int \frac{3x^2-2x+16}{(2-x)(x^2+4)}\,dx$

$$\frac{A}{(2-x)}+\frac{Bx+C}{x^2+4}$$

$A(x^2+4)+(2-x)(Bx+C)=3x^2-2x+16$

let $x=2$ let $x=0$

$8A=12-4+16$ $3(4)+2C=16$

$8A=24$ $12+2C=16$

$A=3$ $2C=4$

$C=2$

$Ax^2-Bx^2=3x^2$ or let $x=1$

$A-B=3$

$3-B=3$

$-B=0$

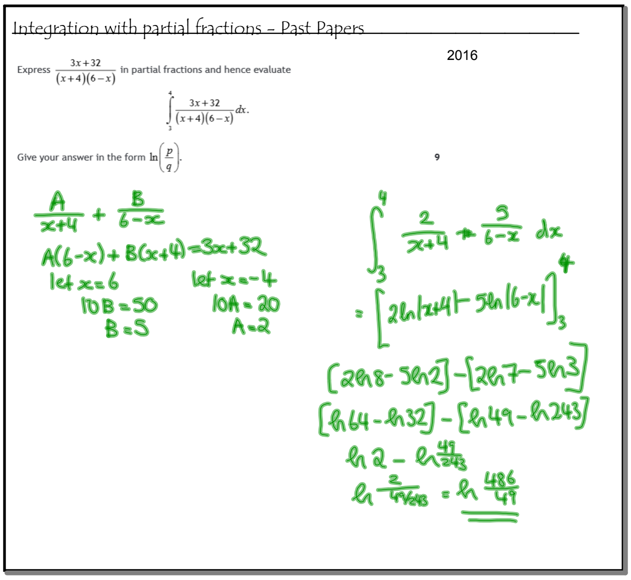
$B=0$

$$\frac{2}{2^2+x^2}$$

$$\int \frac{3}{2-x}+\frac{2}{x^2+4}\,dx$$

$-3\ln|2-x|+\tan^{-1}\left(\frac{x}{2}\right)+C$

Dec 6-14:52



Integration with partial fractions – Past Papers

2016

Express $\frac{3x+32}{(x+4)(6-x)}$ in partial fractions and hence evaluate

$$\int_3^4 \frac{3x+32}{(x+4)(6-x)}\,dx.$$

Give your answer in the form $\ln\left(\frac{p}{q}\right)$. 9

$$\frac{A}{x+4}+\frac{B}{6-x}$$

$A(6-x)+B(x+4)=3x+32$

let $x=6$ let $x=-4$

$10B=50$ $10A=20$

$B=5$ $A=2$

$$\int_3^4 \frac{2}{x+4} + \frac{5}{6-x}\,dx$$

$= \left[2\ln|x+4| - 5\ln|6-x|\right]_3^4$

$[2\ln 8 - 5\ln 2] - [2\ln 7 - 5\ln 3]$

$[\ln 64 - \ln 32] - [\ln 49 - \ln 243]$

$\ln 2 - \ln\frac{49}{243}$

$\ln\frac{2}{49/243} = \ln\frac{486}{49}$

Dec 7-08:19

Integration with partial fractions – Past Papers

2015

Find $\int \frac{2x^3-x-1}{(x-3)(x^2+1)}\,dx,\ x > 3.$ **9**

$$\int \frac{2x^3-x-1}{x^3+x-3x^2-3}\,dx$$

$$x^3-3x^2+x-3 \overset{2}{\overline{\big)\,2x^3+0x^2-x-1}}$$
$$-\ \ 2x^3-6x^2+2x-6$$
$$6x^2-3x+5$$

$$\int 2 + \underbrace{\frac{6x^2-3x+5}{(x-3)(x^2+1)}}$$

$$\frac{A}{x-3} + \frac{Bx+C}{x^2+1}$$

$A(x^2+1)+(Bx+C)(x-3) = 6x^2-3x+5$

Let $x=3$
$10A=50$
$A=5$

let $x=0$
$5-3C=5$
$C=0$

let $x=1$
$10-2B=8$
$-2B=-2$
$B=1$

$$\int 2 + \frac{5}{x-3} + \underbrace{\frac{x}{x^2+1}}\,dx$$

$u=x^2+1$
$\frac{du}{dx}=2x$
$\frac{1}{2}du = x\,dx$
$\frac{1}{2}\int \frac{1}{u}\,du$

$$\underline{2x+5\ln|x-3|+\frac{1}{2}\ln|x^2+1|+C}$$

Dec 7-08:21

Integration with partial fractions – Past Papers

Express $\frac{13-x}{x^2+4x-5}$ in partial fractions and hence obtain

$$\int \frac{13-x}{x^2+4x-5}\,dx.$$

2011

5

$$= \int \frac{13-x}{(x+5)(x-1)}\,dx$$

$$\frac{A}{x+5} + \frac{B}{x-1}$$

$A(x-1)+B(x+5) = 13-x$

let $x=1$
$6B=12$
$B=2$

let $x=-5$
$-6A=18$
$A=-3$

$$\int \frac{-3}{x+5} + \frac{2}{x-1}\,dx$$

$$= \underline{-3\ln|x+5|+2\ln|x-1|+C}$$

Dec 7-08:23

Prelim Revision

Integrate $\frac{\sec^2 3x}{1+\tan 3x}$ with respect to x.

2013

$$\int \frac{\sec^2 3x}{1+\tan 3x}\,dx$$

let $u = 1+\tan 3x$
$\frac{du}{dx} = 3\sec^2 3x$
$du = 3\sec^2 3x\,dx$
$\frac{1}{3}du = \sec^2 3x\,dx$

$$= \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \underline{\frac{1}{3}\ln|1+\tan 3x| + C}$$

Dec 7-08:22

2010

Differentiate the following functions.

(a) $f(x) = e^x \sin x^2.$ **3**

(b) $g(x) = \frac{x^3}{(1+\tan x)}.$ **3**

(a) $e^x(2x\cos x^2) + \sin x^2(e^x)$
$= \underline{e^x\left[2x\cos x^2 + \sin x^2\right]}$

(b) $\frac{(1+\tan x)3x^2 - x^3(\sec^2 x)}{(1+\tan x)^2}$

$\frac{3x^2+3x^2\tan x - x^3\sec^2 x}{(1+\tan x)^2}$

Dec 21-08:09

The height of a cube is increasing at the rate of 5 cm s^{-1}.

Find the rate of increase of the volume when the height of the cube is 3 cm. **4**

2016

$V = h^3$
$\frac{dV}{dh} = 3h^2$

$\frac{dh}{dt} = 5$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = 3h^2 \times 5 = 15h^2$ at $h=3$
$= 15\times 3^2 = \underline{135\text{cm}^3/\text{sec}}$

Dec 21-10:28

(a) Use the substitution $t = x^4$ to obtain $\int \frac{x^3}{1+x^8}\,dx.$ **3**

(b) Integrate $x^2 \ln x$ with respect to x. **4**

2010

Dec 21-08:10

2011

A curve is defined by the equation $y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$ for $x < 1$.

Calculate the gradient of the curve when $x = 0$. **4**

$$y = \frac{e^{\sin x}(2+x)^3}{(1-x)^{1/2}}$$

$$\frac{dy}{dx} = \frac{(1-x)^{\frac{1}{2}}\left[e^{\sin x}\left(3(2+x)^2\right) + (2+x)^3.\cos x e^{\sin x}\right] - e^{\sin x}(2+x)^3\left[-\frac{1}{2}(1-x)^{-\frac{1}{2}}\right]}{1-x}$$

$\frac{dy}{dx} = \underline{\underline{24}}$ at $x=0$

Dec 21-08:12

2010

Given $y = t^3 - \frac{5}{2}t^2$ and $x = \sqrt{t}$ for $t > 0$, use parametric differentiation to express $\frac{dy}{dx}$ in terms of t in simplified form. **4**

Show that $\frac{d^2y}{dx^2} = at^2 + bt$, determining the values of the constants a and b. **3**

Obtain an equation for the tangent to the curve which passes through the point of inflexion. **3**

Dec 21-08:16

2008

Use integration by parts to obtain $\int 8x^2 \sin 4x \, dx$. **5**

Dec 21-08:12

2013

A curve has equation

$$x^2 + 4xy + y^2 + 11 = 0.$$

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-2, 3)$. **6**

Dec 21-08:14

2014

Use the substitution $x = \tan\theta$ to determine the exact value of

$$\int_0^1 \frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}.$$

6

Dec 7-08:22

Dec 21-10:26