Logarithmic Differentiation

When a function involves the product or quotient of powers or roots, logarithmic differentiation can be used to find the derivative of the function.

E.g. 3^x or $y = 2^{x+3}$

We take logarithms to simplify the differentiation.

Logarithmic Differentiation

Examples:

1. $y = 2^{x+3}$

$$\ln y = \ln 2^{x+3}$$
$$\ln y = (x+3)\ln 2$$
$$\ln y = \ln 2 . x + 3\ln 2$$
$$\frac{1}{y}\, dy/dx = \ln 2$$
$$dy/dx = y \ln 2$$
$$dy/dx = 2^{x+3} . \ln 2$$

Logarithmic Differentiation

Examples:

2. $y = \dfrac{x^3}{\sqrt{x+4}} = \dfrac{x^3}{(x+4)^{\frac{1}{2}}}$

$$\ln y = \ln \frac{x^3}{(x+4)^{\frac{1}{2}}}$$
$$\ln y = \ln x^3 - \ln (x+4)^{\frac{1}{2}}$$
$$\ln y = 3\ln x - \tfrac{1}{2}\ln(x+4)$$
$$\frac{1}{y}\, dy/dx = \frac{3}{x} - \frac{1}{2(x+4)} . 1$$
$$\frac{1}{y}\, dy/dx = \frac{6(x+4) - x}{2x(x+4)} = \frac{5x+24}{2x(x+4)}$$
$$dy/dx = y . \frac{5x+24}{2x(x+4)}$$
$$dy/dx = \frac{x^3}{(x+4)^{\frac{1}{2}}} . \frac{5x+24}{2x(x+4)}$$
$$= \frac{x^3(5x+24)}{2x(x+4)^{3/2}}$$
$$dy/dx = \frac{x^2(5x+24)}{2(x+4)^{3/2}}$$

Logarithmic Differentiation

Examples:

3. $y = \dfrac{xe^{x^2}}{\sqrt{\sin x}} = \dfrac{xe^{x^2}}{(\sin x)^{\frac{1}{2}}}$

$$\ln y = \ln \left(\frac{xe^{x^2}}{(\sin x)^{\frac{1}{2}}}\right)$$
$$\ln y = \ln xe^{x^2} - \ln(\sin x)^{\frac{1}{2}}$$
$$\ln y = \ln x + \ln e^{x^2} - \tfrac{1}{2}\ln(\sin x)$$
$$\ln y = \ln x + x^2 \ln e - \tfrac{1}{2}\ln \sin x$$
$$\ln y = \ln x + x^2 - \tfrac{1}{2}\ln \sin x$$
$$\frac{1}{y}\, dy/dx = \frac{1}{x} + 2x - \frac{1}{2\sin x} . \cos x$$
$$\frac{1}{y}\, dy/dx = \frac{1}{x} + 2x - \frac{\cos x}{2\sin x}$$
$$\frac{1}{y}\, dy/dx = \frac{2\sin x + 4x^2\sin x - x\cos x}{2x\sin x}$$
$$dy/dx = \frac{2\sin x + 4x^2\sin x - x\cos x}{2x\sin x} . \frac{xe^{x^2}}{(\sin x)^{\frac{1}{2}}}$$
$$dy/dx = \frac{\cancel{x}e^{x^2}(2\sin x + 4x^2\sin x - x\cos x)}{2\cancel{x}(\sin x)^{3/2}}$$
$$= \frac{e^{x^2}(2\sin x + 4x^2\sin x - x\cos x)}{2(\sin x)^{3/2}}$$

On a suitable domain, a function is defined by $f(x) = \dfrac{e^{x^2-1}}{x^2-1}$.

Find $f'(x)$, simplifying your answer. **3**

$$f'(x) = \frac{(x^2-1)(2xe^{x^2-1}) - e^{x^2-1}(2x)}{(x^2-1)^2}$$
$$= \frac{2x^3e^{x^2-1} - 2xe^{x^2-1} - 2xe^{x^2-1}}{(x^2-1)^2}$$
$$= \frac{2x^3e^{x^2-1} - 4xe^{x^2-1}}{(x^2-1)^2}$$
$$= \frac{2xe^{x^2-1}(x^2-2)}{(x^2-1)^2}$$

$$y = \frac{e^{x^2-1}}{x^2-1}$$
$$\ln y = \ln e^{x^2-1} - \ln(x^2-1)$$
$$\ln y = (x^2-1)\ln e - \ln(x^2-1)$$
$$\ln y = x^2-1 - \ln(x^2-1)$$
$$\frac{1}{y}\, dy/dx = 2x - \frac{2x}{x^2-1}$$
$$\frac{1}{y}\, dy/dx = \frac{2x(x^2-1) - 2x}{x^2-1}$$
$$dy/dx = \frac{2x^3-2x-2x}{x^2-1} . \frac{e^{x^2-1}}{x^2-1}$$
$$dy/dx = \frac{(2x^3-4x)e^{x^2-1}}{(x^2-1)^2}$$
$$= \frac{2xe^{x^2-1}(x^2-2)}{(x^2-1)^2}$$

For $y = 3^{x^2}$, obtain $\dfrac{dy}{dx}$. **3**

$$\ln y = \ln 3^{x^2}$$
$$\ln y = x^2 . \ln 3$$
$$\frac{1}{y}\, dy/dx = \ln 3 . 2x$$
$$dy/dx = \ln 3 . 2x . 3^{x^2}$$

Given $e^y = x^3\cos^2 x$, $x > 0$, show that

$$\frac{dy}{dx} = \frac{a}{x} + b\tan x, \text{ for some constants } a \text{ and } b.$$

State the values of a and b. **3**

$\ln e^y = \ln(x^3\cos^2 x)$

$y\ln e = \ln x^3 + \ln\cos^2 x$

$y = 3\ln x + 2\ln\cos x$

$dy/dx = \frac{3}{x} + \frac{2}{\cos x}\cdot(-\sin x)$

$= \frac{3}{x} - \frac{2\sin x}{\cos x} = \underline{\frac{3}{x} - 2\tan x}$ $a = 3$ $b = -2$

(*a*) Obtain $\frac{dy}{dx}$ when y is defined as a function of x by the equation

$$y + e^y = x^2.$$ **3**

(*b*) Given $f(x) = \sin x\cos^3 x$, obtain $f'(x)$. **3**