Matrices

Numbers are put into arrays to help with multiplication, division etc.

A Matrix (matrices pl.) is a rectangular array of numbers arranged in rows and columns.

They were first introduced to help solve sets of linear equations and are now useful for many applications of Maths including computer generated imaging, electronics, probability & statistics.

Matrices

If there are m rows and n columns in a matrix, then it is an m x n matrix or has order m x n.

$\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$ is a 2 x 2 matrix $\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ is a 3 x 1 matrix

Each number in a matrix is known as an entry or element.

The entry in row i and column j of the matrix A is denoted a_{ij}

E.g.

$A = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$ $a_{12} = 5$

A 'Column Matrix' is one that has only one column and a 'Row Matrix' is one that only has one row.

Matrices

An n x n matrix is known as a Square Matrix.

A Zero Matrix is one in which every element is zero.

Equal Matrices

Two matrices are equal if

(i) They are of the same order

(ii) Their corresponding entries are equal.

E.g. $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ $C = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \end{pmatrix}$

A = B but A ≠ C

Example: Given that $\begin{pmatrix} 2x & 0 \\ -1 & x+y \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ -1 & 1 \end{pmatrix}$, find the values of x and y.

$2x = 6$
$x = 3$

$x + y = 1$
$3 + y = 1$
$y = -2$

Addition and Subtraction of Matrices

If matrices are of the same order they can be added and subtracted.

You can add and subtract the corresponding elements.

Examples:

1. $\begin{pmatrix} 3 & 4 & 5 \\ -2 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -6 \end{pmatrix}$

$= \begin{pmatrix} 5 & 3 & 7 \\ 1 & 2 & -3 \end{pmatrix}$

2. $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 4 & 3 \\ -4 & -1 \end{pmatrix}$

Scalar Multiplication

A scalar is a real number. If k is a real number and A is a matrix, then kA is the matrix obtained by multiplying each entry of A by k.

Examples:

1. If $A = \begin{pmatrix} 3 & 7 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 8 \\ -2 & 5 \end{pmatrix}$ find 4A - 2B

$4A = \begin{pmatrix} 12 & 28 \\ 8 & -4 \end{pmatrix}$ $2B = \begin{pmatrix} 0 & 16 \\ -4 & 10 \end{pmatrix}$

$\begin{pmatrix} 12 & 28 \\ 8 & -4 \end{pmatrix} - \begin{pmatrix} 0 & 16 \\ -4 & 10 \end{pmatrix} = \begin{pmatrix} 12 & 12 \\ 12 & -14 \end{pmatrix}$

2. Given that $2\begin{pmatrix} x & 3 \\ -1 & y \end{pmatrix} - 3\begin{pmatrix} 1 & -z \\ r & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, find the values of x, y, z and r.

$\begin{pmatrix} 2x & 6 \\ -2 & 2y \end{pmatrix} - \begin{pmatrix} 3 & -3z \\ 3r & -12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 2x-3 & 6+3z \\ -2-3r & 2y+12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$2x - 3 = 0$
$x = \frac{3}{2}$

$6 + 3z = 0$
$z = -2$

$-2 - 3r = 0$
$r = \frac{2}{3}$

$2y + 12 = 0$
$y = -6$

Matrices – Transpose

A Transpose of a matrix A′ or A^T, is obtained by interchanging the rows and columns of matrix A.

E.g. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ A′

The Matrix has been reflected in its main diagonal (down from left to right).

The transpose has the following properties:

1. $(A')' = A$
2. $(A + B)' = A' + B'$
3. $(kA)' = kA'$

Symmetric & Skew-Symmetric Matrices

A matrix is said to be symmetric if A′ = A.

E. g. $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{pmatrix}$ then $A' = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{pmatrix}$

A matrix is skew – symmetric if A′ = -A.

E. g. $A = \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & 1 \\ 5 & -1 & 0 \end{pmatrix}$ then $A' = \begin{pmatrix} 0 & -3 & 5 \\ 3 & 0 & -1 \\ -5 & 1 & 0 \end{pmatrix}$

A matrix can **only** be skew-symmetric if the leading diagonal consists of zeros.

Matrix Multiplication

Two matrices A and B may only be multiplied if the number of columns in A is equal to the number of rows in B.

E.g. $(3 \;\; 2 \;\; 1) \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ works but $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \times (3 \;\; 2 \;\; 1)$ doesn't.

rows colums rows columns

$m \times n \quad n \times p$

$\underline{m \times p}$ matrix

Order is very important in Matrix Multiplication. AB ≠ BA except in rare cases. Therefore in general, matrix multiplication is not commutative.

Matrix Multiplication

Matrices are always multipled row by column.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x & y \\ z & r \end{pmatrix} = \begin{pmatrix} ax+bz & ay+br \\ cx+dz & cy+dr \end{pmatrix}$$

Matrix Multiplication

Examples:

1. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

Find the value of AB

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2+2 & 5+6 \\ 6+4 & 15+12 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 4 & 11 \\ 10 & 27 \end{pmatrix}}$$

Matrix Multiplication

Examples:

2. Given $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$2 \times 2 \qquad 2 \times 1$

Find the value of AB

$$AB = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 8+2 \\ 12-4 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

Matrix Multiplication

Examples:

3. Given $P = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & -2 \end{pmatrix}$

$2 \times 2 \qquad 2 \times 3$

Find the value of PQ

$$PQ = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 8+18 & 16+3 & 20-6 \\ -2+12 & -4+2 & -5-4 \end{pmatrix}$$

$$= \begin{pmatrix} 26 & 19 & 14 \\ 10 & -2 & -9 \end{pmatrix}$$

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Multiply out

$$\begin{array}{cc} & \begin{array}{ccc} & C & \end{array} \\ R & \begin{pmatrix} 3 & 1 & 4 \\ 1 & 2 & 5 \end{pmatrix} \end{array} \begin{pmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 7 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 41 & 22 & 19 \\ 41 & 27 & 19 \end{pmatrix}$$

$2 \times 3 \qquad 3 \times 3$

Properties of Matrix Multiplication

1. A(BC) = AB(C) Associative provided the order isn't changed.

2. A(B + C) = AB + AC Distributive over addition

3. ab = 0 => a = 0 or b = 0 in real numbers, this is not the case for Matrices.

Matrix Multiplication

We can find powers of Matrices too.

Example: Given $M = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$

Find M^2, M^3 and M^4

$$M^2 = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3\times3 + -2\times0 & 3\times-2 + -2\times3 \\ 0\times3 + 3\times0 & 0\times-2 + 3\times3 \end{pmatrix}$$

$$M^2 = \begin{pmatrix} 9 & -12 \\ 0 & 9 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 9 & -12 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 3\times9 - 2\times0 & 3\times-12 - 2\times9 \\ 0\times9 + 3\times0 & 0\times-12 + 3\times9 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 27 & -54 \\ 0 & 27 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 27 & -54 \\ 0 & 27 \end{pmatrix} = \begin{pmatrix} 3\times27 - 2\times0 & 3\times-54 - 2\times27 \\ 0\times27 + 3\times0 & 0\times-54 + 3\times27 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 81 & -216 \\ 0 & 81 \end{pmatrix}$$

Identity Matrices

The Identity Matrix is represented by $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ all the entries along the main diagonal are 1's and the rest zeros (It's a square matrix).

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Same Idea

What happens when you multiply a matrix by I?

Given the matrix A, IA = AI = A

A Matrix is said to be **Orthogonal** if A'A = I = AA'

Using Identity Matrices

Examples:

1. Given $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$ and $A^2 = pA + qI$ where $p, q \in Z$

Find the values of p and q.

Hence find the value of A^3 in the form $xA + Iy$ where $x, y \in Z$

$$A^2 = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix}$$

$$pA + qI = \begin{pmatrix} 2p & -1p \\ 3p & 5p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

$$\begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = \begin{pmatrix} 2p+q & -p \\ 3p & 5p+q \end{pmatrix}$$

$3p = 21$, $\underline{p = 7}$ $\quad 2(7) + q = 1$, $\underline{\underline{q = -13}}$ $\quad \boxed{A^2 = 7A - 13I}$

$$A^3 = A(A^2)$$
$$= A(7A - 13I)$$
$$= 7A^2 - 13AI$$
$$= 7(7A - 13I) - 13AI \qquad \boxed{AI = A}$$
$$= 49A - 91I - 13A$$
$$= \underline{\underline{36A - 91I}}$$

Using Identity Matrices

2. Given $A = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

Show that AB = kI for some real number k. ✓

Hence obtain the matrix A^2B

$$AB = \begin{pmatrix} 2\times1 + 2\times2 + -4\times0 & 2\times-1 + 2\times3 + -4\times1 & 2\times0 + 2\times4 + -4\times2 \\ -4\times1 + 2\times2 + -4\times0 & -4\times-1 + 2\times3 + -4\times1 & -4\times0 + 2\times4 + -4\times2 \\ 2\times1 + -1\times2 + 5\times0 & 2\times-1 + -1\times3 + 5\times1 & 2\times0 + -1\times4 + 5\times2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \underline{\underline{6I}}$$

$$A^2B = A(AB) = A(6I) = 6AI = 6A$$

$$= 6\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 12 & 12 & -24 \\ -24 & 12 & -24 \\ 12 & -6 & 30 \end{pmatrix}}}$$

⑤ (a) $A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$

Show that $A^2 = 4A - 7I$

$$A^2 = \begin{pmatrix} 1 & 12 \\ -4 & 1 \end{pmatrix} \qquad 4A - 7I = 4\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 12 \\ -4 & 1 \end{pmatrix}$$

$$A^2 = 4A - 7I$$

(b) $A^3 = pA + qI$

$$A^3 = A(A^2)$$
$$= A(4A - 7I)$$
$$4A^2 - 7AI$$
$$4(4A - 7I) - 7A$$
$$16A - 28I - 7A$$
$$\underline{\underline{9A - 28I}}$$

Determinant of a Square Matrix (2×2)

The determinant of a matrix is a number associated with the matrix and has many important properties and applications.

It is represented by det A or |A|

Determinant of a 2x2 Matrix

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, det A = ad − bc

or this can be written $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Examples:

1. $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 2(4) - 1(3) = \underline{\underline{5}}$

2. $\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 3(1) - 2(-2) = \underline{\underline{7}}$

Determinant of a 2x2 Matrix

3. Find the values of x such that $\begin{vmatrix} x & 2x \\ 2x & x+4 \end{vmatrix} = 1$

$$x(x+4) - 2x(2x) = 1$$
$$x^2 + 4x - 4x^2 = 1$$
$$-3x^2 + 4x = 1$$
$$3x^2 - 4x + 1 = 0$$
$$(3x - 1)(x - 1) = 0$$
$$\underline{\underline{x = \tfrac{1}{3}}}, \ \underline{\underline{x = 1}}$$

Inverse of a 2x2 Matrix

Given two matrices A and B, if AB = I, we say that B is the inverse of A.

The inverse of a square matrix A is A^{-1} which has the property

$AA^{-1} = A^{-1}A = I$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is calculated by $\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where

$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is known as the Adjoint of A or adj.(A)

Inverse of a 2x2 Matrix

Not all Matrices have an inverse. How do you think you could tell?

If det A = 0, then the inverse doesn't exist.

If det A ≠ 0, then the inverse does exist.

If a matrix has an inverse, it is known as invertible (or non-singular), if it doesn't have an inverse then it is non-invertible (or singular).

Inverse of a 2x2 Matrix

Examples:

1. $A = \begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix}$, find A^{-1}

$\det A = 2(11) - 3(7) = 22 - 21 = 1$ ⇒ inverse exists

$A^{-1} = \frac{1}{1}\begin{pmatrix} 11 & -3 \\ -7 & 2 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 11 & -3 \\ -7 & 2 \end{pmatrix}$

2. $P = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$, find P^{-1}

$\det P = 2(3) - (-1)(-4) = 6 - 4 = 2$

$P^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 2 \\ \frac{1}{2} & 1 \end{pmatrix}$

Today we will be learning how to use the inverse of a matrix to solve equations.

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Determine the value of p which would make this matrix singular

$$\begin{pmatrix} p & 3 \\ 8 & 4 \end{pmatrix}$$

If singular det = 0

$ad - bc$

$4p - 24 = 0$

$4p = 24$

$p = 6$

Using the inverse to solve equations

Multiplying by the inverse can help solve equations.

E.g. $AX = Y$

To find X

Premultiply by A^{-1}

$A^{-1}AX = A^{-1}Y$

$IX = A^{-1}Y$

$X = A^{-1}Y$

This is known as Premultiplying.

To find A

Post multiply by X^{-1}

$AX = Y$

$AXX^{-1} = YX^{-1}$

$A = YX^{-1}$

This is known as Postmultiplying.

Using the inverse to solve equations

Examples:

1. (From book): Given A = $\begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix}$

(a) Show that $A^2 = 13A - I$

(b) Hence show that $A^3 = 168A - 13I$ and that $A^{-1} = 13I - A$

(a) $A^2 = \begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix} = \begin{pmatrix} 4+21 & 6+33 \\ 14+77 & 21+121 \end{pmatrix} = \begin{pmatrix} 25 & 39 \\ 91 & 142 \end{pmatrix}$

$13A = 13\begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix} = \begin{pmatrix} 26 & 39 \\ 91 & 143 \end{pmatrix}$

$13A - I = \begin{pmatrix} 25 & 39 \\ 91 & 142 \end{pmatrix} = A^2$

(b) $A^3 = A(A^2)$

$\Rightarrow A(13A - I)$

$= 13A^2 - AI$

$= 13(13A - I) - A = 169A - 13I - A$

$= 168A - 13I$

* $A^{-1} = 13I - A$

We know $A^2 = 13A - I$

$A.A = 13A - I$

Premultiply by A^{-1}

$A^{-1}AA = A^{-1}(13A - I)$

$IA = 13A^{-1}A - A^{-1}$

$A = 13I - A^{-1}$

$A^{-1} = 13I - A$

Using the inverse to solve equations

Examples:

2. Use matrices to solve the set of equations

3x + y = -1 and 4x + 3y = 2

$3x + y = 1$

$4x + 3y = 2$

$\begin{pmatrix} 3x + y \\ 4x + 3y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\times^{-1}$ $\quad \times^{-1}$

$\underbrace{\frac{1}{5}\begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}}_{I}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3(1) - 1(2) \\ -4(1) + 3(2) \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -5 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$x = -1$, $y = 2$

Properties of the Inverse and Determinant

Given that A = $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$, show that

a) $A^{-1}B^{-1} = (BA)^{-1}$

$A^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$B^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 4 & -7 \end{pmatrix}$

$A^{-1}B^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 4 & -7 \end{pmatrix} = \begin{pmatrix} -23 & 41 \\ 9 & -16 \end{pmatrix}$

$BA = \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 16 & 41 \\ 9 & 23 \end{pmatrix}$

$(BA)^{-1} = \frac{1}{-1}\begin{pmatrix} 23 & -41 \\ -9 & 16 \end{pmatrix} = \begin{pmatrix} -23 & 41 \\ 9 & -16 \end{pmatrix}$

$A^{-1}B^{-1} = (BA)^{-1}$

$BAA^{-1}B^{-1} = BA(BA)^{-1}$

$BIB^{-1} = I$

$BB^{-1} = I$

$I = I$

Properties of the Inverse and Determinant

Given that A = $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$, show that

b) $(AB)^{-1} = B^{-1}A^{-1}$

$AB = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 34 & 9 \\ 19 & 5 \end{pmatrix}$

$(AB)^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -9 \\ -19 & 34 \end{pmatrix} = \begin{pmatrix} -5 & 9 \\ 19 & -34 \end{pmatrix}$

$B^{-1}A^{-1} = \begin{pmatrix} -1 & 2 \\ 4 & -7 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 9 \\ 19 & -34 \end{pmatrix}$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

Properties of the Inverse and Determinant

Given that A = $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$, show that

c) det(AB) = det A × det B or |AB| = |A||B|

$\det AB = \frac{1}{34(5) - 9(19)} = \frac{1}{-1} = -1$

$\det A = 1 \qquad \det B = -1$

$1 \times -1 = -1$

$\Rightarrow \det AB = \det A \times \det B$

Determinant of a 3x3 Matrix

Given the matrix A = $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, the determinant can be found by

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

|A| = a(ei – fh) – b(di – fg) + c(dh – eg)

I remember this as right, outsides, left

and + – +

Determinant of a 3x3 Matrix

Examples: Calculate the determinant of the following

(i) $A=\begin{pmatrix}1 & 2 & 1\\ 4 & -2 & 3\\ 2 & 1 & -1\end{pmatrix}$

$|A| = 1\begin{vmatrix}-2 & 3\\ 1 & -1\end{vmatrix} - 2\begin{vmatrix}4 & 3\\ 2 & -1\end{vmatrix} + 1\begin{vmatrix}4 & -2\\ 2 & 1\end{vmatrix}$

$= 1((-2\times-1)-(3\times1)) - 2((4\times-1)-(3\times2)) + 1((4\times1)-(2\times2))$

$= 1(-1) - 2(-10) + 1(8) = \underline{\underline{27}}$

(ii) $B=\begin{pmatrix}3 & -2 & 0\\ 1 & 4 & 5\\ 0 & -1 & 2\end{pmatrix}$

$|B| = 3\begin{vmatrix}4 & 5\\ -1 & 2\end{vmatrix} - (-2)\begin{vmatrix}1 & 5\\ 0 & 2\end{vmatrix} + 0\begin{vmatrix}1 & 4\\ 0 & -1\end{vmatrix}$

$= 3(8+5) + 2(2-0) + 0$

$= 39+4$

$= \underline{\underline{43}}$

Determinant of a 3x3 Matrix

3. Find the values of k for which the det A = 0

$A=\begin{pmatrix}k & 1 & 1\\ 2 & -2 & 3\\ -1 & -1 & -k\end{pmatrix}$

$|A| = k\begin{vmatrix}-2 & 3\\ -1 & -k\end{vmatrix} - 1\begin{vmatrix}2 & 3\\ -1 & -k\end{vmatrix} + 1\begin{vmatrix}2 & -2\\ -1 & -1\end{vmatrix}$

$= k(2k+3) - 1(-2k+3) + 1(-2-2) = 0$

$= 2k^2+3k+2k-3-4=0$

$= 2k^2+5k-7=0$

$(2k+7)(k-1)=0$

$\underline{\underline{k=\frac{-7}{2},\ k=1}}$

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Calculate the determinant of A given

$A = \begin{pmatrix}1 & 2 & 3\\ 0 & -4 & 1\\ 0 & 3 & -1\end{pmatrix}$

$\det A = 1\begin{vmatrix}-4 & 1\\ 3 & -1\end{vmatrix} - 2\begin{vmatrix}0 & 1\\ 0 & -1\end{vmatrix} + 3\begin{vmatrix}0 & -4\\ 0 & 3\end{vmatrix}$

$= 1(4-3) - 2(0-0) + 3(0-0)$

$= \underline{\underline{1}}$

Inverse of a 3x3 Matrix

We can get the inverse of a 3 x 3 matrix by carrying out row operations. Row operations can include interchanging rows, multiplying by a scaler and adding/sutracting rows from each other.

We know that A x A^{-1} = I, so we use the row operations to convert A into I and record the result to get A^{-1}.

Start at the top left entry and work under the diagonal to begin with.

Inverse of a 3x3 Matrix

Examples:

1. Find A^{-1}, given A = $\begin{pmatrix}1 & 1 & 1\\ 2 & -3 & -1\\ 5 & 2 & 3\end{pmatrix}$

$r_2-2r_1 \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 2 & -3 & -1 & 0 & 1 & 0\\ 5 & 2 & 3 & 0 & 0 & 1\end{array}\right)$

make into zeros

$r_3-5r_1 \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & -5 & -3 & -2 & 1 & 0\\ 5 & 2 & 3 & 0 & 0 & 1\end{array}\right)$

$r_2 \div -5 \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & -5 & -3 & -2 & 1 & 0\\ 0 & -3 & -2 & -5 & 0 & 1\end{array}\right)$

$r_3+3r_2 \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0\\ 0 & -3 & -2 & -5 & 0 & 1\end{array}\right)$

$-5r_3 \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0\\ 0 & 0 & -\frac{1}{5} & -\frac{19}{5} & -\frac{3}{5} & 1\end{array}\right)$

$r_2-\frac{3}{5}r_3 \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0\\ 0 & 0 & 1 & 19 & 3 & -5\end{array}\right)$

$r_1-r_2-r_3 \left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & -11 & -2 & 3\\ 0 & 0 & 1 & 19 & 3 & -5\end{array}\right)$

$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -7 & -1 & 2\\ 0 & 1 & 0 & -11 & -2 & 3\\ 0 & 0 & 1 & 19 & 3 & -5\end{array}\right)$

$A^{-1} = \underline{\underline{\begin{pmatrix}-7 & -1 & 2\\ -11 & -2 & 3\\ 19 & 3 & -5\end{pmatrix}}}$

Inverse of a 3x3 Matrix

Examples:

2. Find B^{-1}, given B = $\begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 2 & 0 & 1\end{pmatrix}$

$B^{-1} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{4} & 0\\ 0 & \frac{1}{2} & 0\\ -1 & \frac{1}{2} & 1\end{pmatrix}$

$\begin{pmatrix}7 & 2 & 1\\ 0 & 3 & -1\\ -3 & 4 & -2\end{pmatrix}$

Inverse of a 3x3 Matrix

Examples:

3. Find C^{-1}, given $C = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$

Matrix Transformations

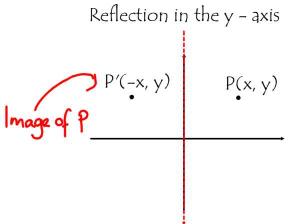
Matrices can be used to transform coordinates and objects on a Plane.

The coordinates can be written in matrix form and then can be multiplied by a matrix or scalar for Rotation, Reflection or Dilation (Scaling).

If under a transformation a point is its own image, it is then called invariant.

Matrix Transformations

Reflection in the y – axis



We write P in matrix form $\begin{pmatrix} x \\ y \end{pmatrix}$.

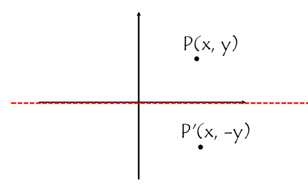
What can we multiply P by to get $\begin{pmatrix} -x \\ y \end{pmatrix}$?

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the matrix associated with reflection in the y – axis.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Matrix Transformations

Reflection in the x – axis

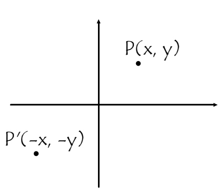


What can we multiply P by to get $\begin{pmatrix} x \\ -y \end{pmatrix}$?

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with reflection in the x – axis.

Matrix Transformations

Reflection in the origin

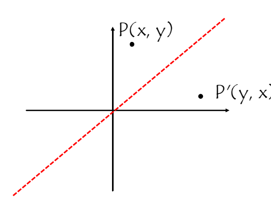


What can we multiply P by to get $\begin{pmatrix} -x \\ -y \end{pmatrix}$?

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with reflection in the origin.

Matrix Transformations

Reflection in the line y = x



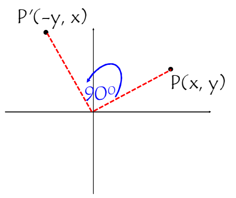
What can we multiply P by to get $\begin{pmatrix} y \\ x \end{pmatrix}$?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the matrix associated with reflection in the line y = x.

Matrix Transformations

Points can be rotated by an angle around the origin.



What could we multiply P by?

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-y\\ x\end{pmatrix}$$

The matrix associated with rotation is R = $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

where points are rotated anti-clockwise around the origin.

Matrix Transformations

Example: Calculate the coordinates of the image of the point (3, 1) under a 60° rotation about the origin

$$\begin{pmatrix}\cos\frac{\pi}{3} & -\sin\frac{\pi}{3}\\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}-\frac{\sqrt{3}}{2}\\ \frac{3\sqrt{3}}{2}+\frac{1}{2}\end{pmatrix}$$

$$P'=\left(\frac{3-\sqrt{3}}{2},\frac{3\sqrt{3}+1}{2}\right)$$

Matrix Transformations

If we use R to rotate a point anti-clockwise by θ, then we rotate the point back in a clockwise direction using R^{-1}.

$$R^{-1}=\frac{1}{\cos^2\theta+\sin^2\theta}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

We can see that $R^{-1} = R^T$, therefore $R \times R^T = I$ which means R is orthogonal.

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Write down the 2 x 2 matrix associated with a clockwise rotation of 30° about the origin.

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$$

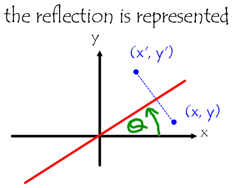
The Reflection Matrix

The Reflection Matrix $\boxed{S=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}}$

A = $\begin{pmatrix}x\\ y\end{pmatrix}$ represents the point (x, y)

Pre-multiplying A by the reflection matrix has the effect of reflecting the point (x, y) in the line through the origin which makes an angle of θ° where -90 ≤ θ° ≤ 90.

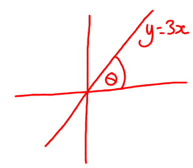
The point gained from the reflection is represented by (x′, y′).



The Reflection Matrix

Example:

Find the coordinates of the point (-3, 1) when reflected in the line y = 3x



$\tan^{-1}(3) = 71.6°$

$$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$$

$$\begin{pmatrix}\cos 143.2° & \sin 143.2°\\ \sin 143.2° & -\cos 143.2°\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-0.8 & 0.599\\ 0.599 & 0.8\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-0.8(-3)+0.599(1)\\ 0.599(-3)+0.8(1)\end{pmatrix}=\begin{pmatrix}3\\ -1\end{pmatrix}$$

(3, -1)

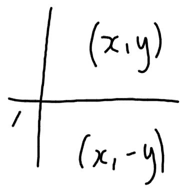
Composite Transformations

A composite transformation is a matrix that undergoes more than one transformation.

Let's say we want to reflect P(x, y) in the x – axis and then reflect it in the line y = x.

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Composite Transformation $M_2 \times M_1 \times \begin{pmatrix} x \\ y \end{pmatrix}$

Composite Transformations

Example: The point P(x, y) is given an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin and the image is reflected in the x – axis

(a) Find the matrix associated with this composite function

$$M_1 = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_2 \times M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(b) Find the coordinates of the image of P(x, y) under this composite transformation

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix} = (-y, -x)$$

Dilation (Scaling)

Think about what matrix you could multiply a point (x, y) by to scale it.

$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ premultiplies by the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$

The point (x, y) will scale x by λ and y by μ

Dilation (Scaling)

Examples:

1. The point (-6, 8) is scaled to give (4, 24). Find the scaling matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 24 \end{pmatrix}$$

$8\mu = 24$, $\mu = 3$

$-6\lambda = 4$, $\lambda = -\frac{2}{3}$

$$\begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & 3 \end{pmatrix}$$

2. Find the effect of (-3, -9) using the scaling matrix $\begin{pmatrix} -2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$

$$\begin{pmatrix} -2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -3 \\ -9 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

(6, -6)

General Linear transformation of a line

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can transform sets of points on a line b, c ≠ 0

Example:

Find the 2 x 2 matrix which would transform (-3, 2) to (-8, -13) and (5, 4) to (6, 7)