Starter 5.12.2016

Use Gaussian elimination to show that the set of equations

$$x - y + z = 1$$
$$x + y + 2z = 0$$
$$2x - y + az = 2$$

has a unique solution when $a \neq 2.5$. 5 marks

Explain what happens when $a = 2.5$. 1 mark

Obtain the solution when $a = 3$. 1 mark

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & a & 2 \end{array}\right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 2 & -1 & a & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & a-2 & 0 \end{array}\right) \xrightarrow[r_3 - \frac{1}{2}r_2]{} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & a-2-\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$\left(a - 2 - \frac{1}{2}\right)z = \frac{1}{2}$

$\left(a - \frac{5}{2}\right)z = \frac{1}{2}$

$z = \dfrac{0.5}{a - 2.5}$

$2y + z = -1$

$2y + \dfrac{0.5}{a - 2.5} = -1$

$y = \dfrac{-a+2}{2a-5}$

$x - \left(\dfrac{-a+2}{2a-5}\right) + \dfrac{1}{2a-5} = 1$

$x = \dfrac{a-4}{2a-5}$

$\Rightarrow$ There is a unique solution when $a \neq 2.5$

If $a = 2.5$, a solution doesn't exist because you cannot $\div 0$.

If $a = 3 \Rightarrow x = \frac{-1}{1} = -1$, $y = -1$ $z = 1$

Sequences

A sequence is an ordered list of terms. Each number in the sequence is known as a term or element.

The nth term or general term is denoted by u_n.

A finite squence is one that has a last term.

An infinite squence is one which continues indefinitely. E.g. natural numbers listed in order or prime numbers listed in order.

Sequences

n denotes the position of each term.

For example:

$\frac{1}{n}$ denotes the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$

2n + 1 denotes the sequence 3, 5, 7, 9, 11...

If we have a formula for the general or nth term, it is possible to find any specific term in the sequence.

Sequences

Examples:

1. Given $u_n = \dfrac{2n}{n+1}$, find the first 4 terms of the sequence

$u_1 = \frac{2(1)}{1+1} = 1$ $u_2 = \frac{2(2)}{2+1} = \frac{4}{3}$ $u_3 = \frac{2(3)}{3+1} = \frac{6}{4} = \frac{3}{2}$

$u_4 = \frac{2(4)}{4+1} = \frac{8}{5}$

2. Find the 4th term of the recurrence relation

$u_{n+1} = 0.5u_n + 3$ where $u_1 = 2$

$u_2 = 0.5(2) + 3 = 4$

$u_3 = 0.5(4) + 3 = 5$

$u_4 = 0.5(5) + 3 = 5.5$

Arithmetic Sequences

An arithmetic sequence is one which increases or decreases by a constant amount. E.g. 2, 5, 8, 11...

The sequence takes the form a, a + d, a + 2d, a + 3d ... where 'a' is the first term and 'd' is the common difference (the difference between any two consecutive terms in the sequence).

$u_1 = a$

$u_2 = a + d$

$u_3 = a + 2d$

$\vdots$

$u_n = a + (n-1)d$

$\underset{u_1}{3}, \underset{u_2}{7}, \underset{u_3}{11}, \underset{u_4}{15} \ldots$

a, a + d, a + 2d, a + 3d . . . . a + (n – 1)d

$3 + (n-1)4$

$3 + 4n - 4$

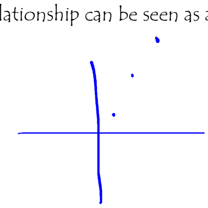
$4n - 1$

General or nth term of an Arithmetic Sequence

Arithmetic Sequences

If the values of the terms of an Arithmetic Sequence are plotted on a graph against the term position, the relationship can be seen as a linear one.

u_1	1
u_2	4
u_3	7
u_4	10
$\vdots$	$\vdots$



Arithmetic Sequences

Examples:

1. Find the 25th term of the arithmetic sequence 2, 6, 10, 14...

$u_n = a + (n-1)d$ or $4n-2$

$u_n = 2 + (n-1)4$

$u_{25} = 2 + (25-1)4 = \underline{\underline{98}}$

$u_1 \quad u_2 \quad u_3$

2. For the sequence 8, 12, 16, 20...

(a) Find the nth term rule

$u_n = a + (n-1)d$

$u_n = 8 + (n-1)4$

$u_n = 8 + 4n - 4$ $\qquad u_n = \underline{\underline{4n+4}}$

(b) The 100th term

$u_{100} = 4(100) + 4 = \underline{\underline{404}}$

Arithmetic Sequences

3. Find a formula for the nth term u_n of the sequence 1, 4, 7, 10...

$u_n = a + (n-1)d$

$u_n = 1 + (n-1)3$

$u_n = \underline{\underline{3n-2}}$

4. The 3rd term of an arithmetic sequence is 40 and the 5th term is 30.

(a) Find the first term and the common difference.

$u_3 = a + (n-1)d = 40$

$u_3 = a + (3-1)d = 40$

$a + 2d = 40$

$u_5 = a + (5-1)d = 30$

$a + 4d = 30$

$-\; a + 2d = 40$

$2d = -10$

$\underline{\underline{d = -5}}$ decreasing sequence

$a + 2(-5) = 40$

$\underline{\underline{a = 50}}$

(b) Find the 15th term of the sequence.

$u_{15} = 50 + (15-1)(-5)$

$= 50 - 70$

$= \underline{\underline{-20}}$

Starter

In the arithmetic sequence beginning 2, 8, 14, 20 ... which term is the first term to exceed 100?

$u_n = a + (n-1)d$

$= 2 + 6(n-1)$

$= 6n - 4$

$100 = 6n - 4$

$104 = 6n$

$n = 17.3$

$n = \underline{\underline{18}}$ So it would be the 18th term.

$u_{18} = 2 + 6(18-1)$

$= 2 + 6(17) = \underline{\underline{104}}$

Finding the sum of an Arithmetic Sequence

Consider the series 1, 2, 3, 4, ... 100 where S is the sum of the series.

If we also look at the reverse of the series 100, 99, 98, ... 1 it will have the same sum.

Therefore if we add the series together, we get

S + S = 2S = 101 + 101 + 101 + ... + 101

This means that 2S = 100 x 101, so what is the value of S?

$S = \underline{\underline{5050}}$

Finding the sum of an Arithmetic Sequence

We can use a similar method to find the sum of an arithmetic sequence.

Let S_n = the sum of the first n terms

=> $S_n = u_1 + u_2 + u_3 + ... + u_n$

For the arithmetic sequence $u_1 = a$, $u_2 = a + d$, $u_3 = a + 2d$ ...

=> $S_n = a + a+d + a+2d + \ldots + a+(n-1)d$

$S_n = a+(n-1)d + a+(n-2)d + a+(n-3)d + \ldots + a$

$2S_n = 2a+(n-1)d + 2a+d+(n-2)d + 2a+2d+(n-3)d + \ldots + 2a+(n-1)d$

$2a+d+nd-2d$

$2a+nd-d$

$2a+(n-1)d + 2a+d(n-1) + \ldots + 2a+(n-1)d$

$2S_n = n(2a+(n-1)d)$

$\underline{\underline{S_n = \frac{n}{2}(2a+(n-1)d)}}$ Sum of the first n terms of an arithmetic sequence. (On formulae Sheet)

Finding the sum of an Arithmetic Sequence

Examples:

1. Find the sum of the first 15 terms of the arithmetic sequence

3, 8, 13, 18...

$S_n = \frac{n}{2}(2a + (n-1)d)$ $\qquad a = 3 \quad n = 15 \quad d = 5$

$S_{15} = \frac{15}{2}(2(3) + (15-1)5)$

$S_{15} = \underline{\underline{570}}$

2. Find the sum of the arithmetic series 12, 19, 26, 33, ... 285

last term Find position

$a + (n-1)d$

$285 = 12 + (n-1)(7)$

$285 = 12 + 7n - 7$

$280 = 7n$

$\underline{n = 40}$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{40} = \frac{40}{2}(2(12) + (40-1)7)$

$S_{40} = \underline{\underline{5940}}$

Finding the sum of an Arithmetic Sequence

Examples:

3. Let u_n denote the nth term of the arithmetic sequence 2, 10, 18, 26 ...

Let $S_n = \sum_{k=1}^{n} u_k$

(a) Find a formula for S_n in terms of n.

(b) Find the least value of n for which $S_n > 1000$.

(c) Evaluate $\sum_{k=20}^{40} u_k$

(a) $S_n = \frac{n}{2}(2a+(n-1)d)$

$a = 2 \quad d = 8$

$S_n = \frac{n}{2}(2(2)+(n-1)8)$

$= 2n + 4n(n-1)$

$= 2n + 4n^2 - 4n$

$= \underline{\underline{4n^2 - 2n}}$

(b) $4n^2 - 2n > 1000$

$4n^2 - 2n - 1000 > 0$

$2n^2 - n - 500 > 0$

$n = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-500)}}{2(2)}$

$n = \cancel{-15.6}$ and 16.1

It would be the 17^{th} term

Check $S_{16} = 4(16)^2 - 2(16) = 992 < 1000$

$S_{17} = 4(17)^2 - 2(17) = 1122 > 1000$

(c) $\sum_{k=20}^{40} u_k$

Sum of the terms from 20 to 40

$S_{40} = 4(40)^2 - 2(40) = 6320$

$S_{19} = 4(19)^2 - 2(19) = 1406$

$S_{40} - S_{19} = 6320 - 1406 = \underline{\underline{4914}}$

Starter

The sum of the first twenty terms of an arithmetic sequence is 320.

The twenty-first term is 37.

What is the sum of the first ten terms? **5**

$S_{20} = 320 \quad u_{21} = 37$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{20} = 10(2a + (20-1)d)$

$320 = 20a + 190d$

$37 = a + 20d$

$\ominus \; 20a + 190d = 320$

$20a + 400d = 740$

$-210d = -420$

$\underline{\underline{d = 2}}$

$a + 40 = 37$

$\underline{\underline{a = -3}}$

$S_{10} = \frac{10}{2}(2(-3) + (10-1)2)$

$= 5(-6 + 18)$

$= \underline{\underline{60}}$

Finding the sum of an Arithmetic Sequence

The terms of a sequence are given by $U_k = 11 - 2k$, $k \geq 1$

(a) Obtain a formula for S_n where $S_n = \sum_{k=1}^{n} u_k$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$U_1 = 11 - 2(1) = 9 \quad a = 9$

$U_2 = 11 - 2(2) = 7 \quad d = -2$

$S_n = \frac{n}{2}(2(9) + (n-1)(-2))$

$= \frac{n}{2}(18 - 2n + 2)$

$= \underline{\underline{10n - n^2}}$

(b) Find the values of n for which $S_n = 21$

$21 = 10n - n^2$

$n^2 - 10n + 21 = 0$

$(n-7)(n-3) = 0$

$\underline{\underline{n = 7}}$ or $\underline{\underline{n = 3}}$

Finding the sum of an Arithmetic Sequence

5. The sum S_n of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = 8n - n^2$, $n \geq 1$.

(a) Calculate the values of u_1, u_2, u_3 and state what type of sequence it is.

(b) Obtain a simplified formula for u_n in terms of n.

(a) $S_1 = 8(1) - 1^2 = 7$

$S_2 = 8(2) - 2^2 = 12$

$S_3 = 8(3) - 3^2 = 15$

$u_1 = S_1 = 7$

$u_2 = 12 - 7 = 5$

$u_3 = 15 - 12 = 3$

$a = 7$

$d = -2$

It is a decreasing arithmetic sequence.

(b) $u_n = a + (n-1)d$

$= 7 + (n-1)-2$

$= 7 - 2n + 2$

$= \underline{\underline{9 - 2n}}$

Geometric Sequences

Consider the sequence 1, 2, 4, 8, 16, ...

The first term is 1 and the terms are multiplied by 2 each time. This is a geometric sequence with the first term a = 1 and common ratio r = 2.

The general geometric sequence

The nth term is denoted by u_n

$u_1 = a \quad u_2 = a \times r \quad u_3 = a \times r^2 \quad u_4 = a \times r^3$ and so on

In general $u_n = ar^{n-1}$. We can find particular terms with this formula.

Note: The general sequence is also defined by the recurrence relation $u_{n+1} = ru_n$ with the first term $u_1 = a$

Geometric Sequences

Examples:

1. Find the 10th term of the sequence 3, 12, 48, 192, ...

$u_n = ar^{n-1}$

$u_{10} = 3(4)^{10-1} = 3(4^9) = \underline{786432}$

2. Find the 5th term of the sequence which is of positive terms with 3rd term 18 and 7th term 1458

$u_3 = 18$

$18 = ar^2$

$\frac{18}{r^2} = a$

$u_7 = 1458$

$1458 = ar^6$

$1458 = \left(\frac{18}{r^2}\right) r^{4}$

$18r^4 = 1458$

$r^4 = 81$

$r = \pm\sqrt[4]{81} = \pm 3$

$\underline{\underline{r = 3}}$

$\frac{18}{3^2} = a$

$\underline{\underline{a = 2}}$

$u_5 = ar^4$

$= 2(3^4) = \underline{\underline{162}}$

Starter

Matrices A and B are defined by $A = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}$.

(a) Find A^2. 1

(b) Find the value of p for which A^2 is singular. 2

(a) $A^2 = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 16-2p & 5p \\ -10 & -2p+1 \end{pmatrix}$

(b) Singular $\Rightarrow$ No inverse $\Rightarrow$ det $A^2 = 0$

$(16-2p)(-2p+1) - (5p)(-10) = 0$

$= -32p + 16 + 4p^2 - 2p + 50p = 0$

$4p^2 + 16p + 16 = 0$

$p^2 + 4p + 4 = 0$

$(p+2)(p+2) = 0$

$\underline{\underline{p = -2}}$

Sum of a Geometric Series

$S_n = u_1 + u_2 + u_3 + \ldots + u_n$

$u_1 = a \quad u_2 = ar \quad u_3 = ar^2 \; . \; . \; . \; u_n = ar^{n-1}$

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$\underline{\underline{S_n = \dfrac{a(1-r^n)}{1-r}}} \quad r \neq 1$

If $r = 1$

$\underline{\underline{S_n = na}}$

Sum of a Geometric Series

Examples:

1. Find the sum of the first 9 terms of the geometric series 4, 8, 16, 32, ...

$S_n = \dfrac{a(1-r^n)}{1-r} \quad a = 4 \quad r = 2$

$S_9 = \dfrac{4(1-2^9)}{1-2} = \dfrac{4(-511)}{-1} = \underline{\underline{2044}}$

Sum of a Geometric Series

Examples:

2. Evaluate $\sum_{k=1}^{20} (0.9)^k$ and give your answer to 3 d.p.

$u_1 = 0.9^1 = 0.9 \quad u_2 = 0.9^2 = 0.81 \quad u_3 = 0.9^3 = 0.729$

$a = 0.9 \quad r = 0.9$

$S_{20} = \dfrac{0.9(1-0.9^{20})}{1-0.9} = \underline{\underline{7.906}}$ (3 d.p.)

3. Evaluate the sum of the geometric series 4, 20, 100, ..., 62500

$u_n = ar^{n-1}$

$62500 = 4(5^{n-1})$

$15625 = 5^{n-1}$

$\log_e 15625 = \log_e 5^{n-1}$

$\log_e 15625 = (n-1)\log_e 5$

$\dfrac{\log_e 15625}{\log_e 5} = n-1$

$n = \dfrac{\log_e 15625}{\log_e 5} + 1 = \underline{\underline{7}}$

$S_7 = \dfrac{4(1-5^7)}{1-5} = \underline{\underline{78124}}$

Sum of a Geometric Series

Examples:

4. Find the least number of terms of the geometric series 4, 12, 36, 108... which must be added to give a sum exceeding 1 000 000

$a = 4 \quad r = 3$

$S_n = \dfrac{4(1-3^n)}{1-3} = 1000000$

$= \dfrac{4(1-3^n)}{-2} = 1000\,000$

$-2(1-3^n) = 1000\,000$

$\Rightarrow 1 - 3^n = -500\,000$

$3^n = 500\,001$

$\log_e 3^n = \log_e 500\,001$

$n = \dfrac{\log_e 500001}{\log_e 3} = 11.94$

$n = 12$

$S_{11} = \dfrac{4(1-3^{11})}{1-3} = 354292$

$S_{12} = \dfrac{4(1-3^{12})}{1-3} = 1062\,880$

$S_{12} > 1000000$

$\underline{\underline{\text{12 terms}}}$

Starter

A line 315cm in length is divided into 6 parts such that the length of the parts form a geometric sequence.

Given that the length of the longest part is 32 times larger than that of the shortest part, find the length of the shortest part

$u_6 = 32u_1 \quad S_6 = 315$

$u_6 = ar^5$

$u_6 = 32a$

$32\cancel{a} = \cancel{a}r^5$

$r = \sqrt[5]{32} = 2$

$315 = \dfrac{a(1-r^6)}{1-r}$

$315 - 315r = a - ar^6$

$315 - 315(2) = a - 64a$

$-315 = -63a$

$\underline{\underline{a = 5}}$

Shortest part = $\underline{\underline{5\text{cm}}}$

Limits & Infinite Geometric Series

Consider the sum of the first n terms of the geometric series 1, 2, 4, 8... a = 1 and r = 2

$S_n = \frac{a(1-r^n)}{1-r} = \frac{1(1-2^n)}{1-2}$

$S_n = \frac{1-2^n}{-1} = \underline{\underline{2^n-1}}$

$S_5 = 31$ $S_{10} = 1023$ $S_{20} = 1048575$

The sum of the first n terms increases infinitely without a limit. This is known as a divergent series.

Limits & Infinite Geometric Series

Now consider the sequence $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$

a = 1 , r = $\frac{1}{2}$

$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow \frac{1(1-\frac{1}{2}^n)}{1-\frac{1}{2}} = \frac{1-\frac{1}{2^n}}{\frac{1}{2}}$

$= 2 - \frac{2}{2^n} = 2-2^{1-n}$

or $\underline{\underline{2(1-\frac{1}{2^n})}}$

$S_5 = 1.9375$ $S_{10} = 1.998...$ $S_{20} = 1.999......$

The sum of the first n terms approaches a limit of 2 as n tends to infinity. This is known as a convergent sequence.

We say that this geometric series has a sum to infinity of 2 and write $S_\infty = 2$

Limits & Infinite Geometric Series

How do you think you could determine whether a sequence has a limit?

It is the value of the common ratio 'r' that determines whether or not a geometric series has a sum to infinity.

$-1 < r < 1$ means it has a sum to infinity

Given $S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$ if $-1 < r < 1$, then $r^n \to 0$ as $n \to \infty$

$S_\infty = \frac{a(1-0)}{1-r} = \underline{\underline{\frac{a}{1-r}}}$

Limits & Infinite Geometric Series

Examples:

1. Explain why the geometric series 27, 18, 12, ... has a sum to infinity and find the value of the sum to infinity

$r = \frac{2}{3}$ therefore $-1 < \frac{2}{3} < 1 \Rightarrow$ It has a sum to infinity.

$S_\infty = \frac{a}{1-r} = \frac{27}{1-\frac{2}{3}} = \frac{27}{\frac{1}{3}} = \underline{\underline{81}}$

2. u_1 , u_2 , u_3 , ...is a geometric sequence with $u_1 = 48$ and $\sum_{k=1}^{\infty} u_k = 64$

Find the value of u_4

$a = 48$

$S_\infty = \frac{a}{1-r}$

$64 = \frac{48}{1-r}$

$64 - 64r = 48$

$-64r = -16$

$\underline{\underline{r = \frac{1}{4}}}$

$U_n = ar^{n-1}$

$U_4 = 48(\frac{1}{4})^{4-1}$

$= 48(\frac{1}{64})$

$= \frac{48}{64} = \underline{\frac{3}{4}}$

$S_\infty = 64$

$ar = -6$

$ar^2 = 3$

$r = \frac{-6}{a}$

$a(-\frac{6}{a})^2 = 3$

$\cancel{a}(\frac{36}{a^{\cancel{2}}}) = 3$

$\frac{36}{a} = 3$

$3a = 36$

$a = 12$

$r = \frac{-6}{12} = \frac{-1}{2}$

Starter

16mins

1. Write down and simplify the general term in the expression $\left(\frac{2}{x}+\frac{1}{4x^2}\right)^{10}$.

Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$. **5**

$\binom{10}{r}\left(\frac{2}{x}\right)^{10-r}\left(\frac{1}{4x^2}\right)^r$ ✓

$\binom{10}{r}(2^{10-r})(x^{-10+r})(\frac{1}{4})^r(x^{-2r})$

$\binom{10}{r}(2^{10-r})(2^{-2r})(x^{-10-r})$

$\underline{\underline{\binom{10}{r}(2^{10-3r})(x^{-10-r})}}$ ✓ ✓

$x^{-10-r} = x^{-13}$

$-10-r = 13$

$-r = -3$

$\underline{\underline{r = 3}}$ ✓

$\binom{10}{3}(2)(x^{-13})$

$= \underline{\underline{\frac{240}{x^{13}}}}$ ✓

2. Below is a system of equations:

$$x+2y+3z=3$$
$$2x-y+4z=5$$
$$x-3y+2\lambda z=2$$

Use Gaussian elimination to find the value of λ which leads to redundancy. **4**

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & -1 & 4 & 5 \\ 1 & -3 & 2\lambda & 2 \end{array}\right) \xrightarrow{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -5 & -2 & -1 \\ 1 & -3 & 2\lambda & 2 \end{array}\right)$$

$$\xrightarrow{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -5 & -2 & -1 \\ 0 & -5 & 2\lambda-3 & -1 \end{array}\right) \xrightarrow{r_3-r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -5 & -2 & -1 \\ 0 & 0 & 2\lambda-1 & 0 \end{array}\right)$$

$2\lambda-1=0$

$\lambda=\frac{1}{2}$

$0=0$ Redundancy

Two matrices A and B are given as

$$A=\begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & \alpha \\ 1 & 0 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} 4 & 1 & 2 \\ -1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Find the value of α for which the matrix $3A-2B$ is singular. **5**

Find the term independent of u in the expansion of $\left(\frac{2}{u^3}-3u\right)^8$. **4**

(*a*) Express $z=\frac{4-2i}{3+i}-(1-2i)(3+i)$ in the form $x+iy$ where x and y are real numbers. **3**

(*b*) (i) Show that $-2i$ is a root of the equation $z(z^3+8z^2+36z+32)+128=0$. **2**

(ii) Write down a second root of the equation $z(z^3+8z^2+36z+32)+128=0$. **1**

(iii) Find the other two roots of the equation $z(z^3+8z^2+36z+32)+128=0$. **3**

Use Gaussian elimination to solve the following system of equations

$$x+2y+z=4$$
$$3x+5y+6z=10$$
$$2x+5y-z=10.$$ **5**

Find the solution corresponding to y = -1. **1**

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 5 & 6 & 10 \\ 2 & 5 & -1 & 10 \end{array}\right) \xrightarrow{r_2-3r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 3 & -2 \\ 2 & 5 & -1 & 10 \end{array}\right)$$

$$\xrightarrow{r_3-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 3 & -2 \\ 0 & 1 & -3 & 2 \end{array}\right) \xrightarrow{r_3+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-y+3z=-2$

$1+3z=-2$

$3z=-3$

$y=-1$ $z=-1$

$x+2y+z=4$

$x-2-1=4$

$x=7$

Starter 11.1.2017

The first two terms of a series are $1+\sqrt{5}$ and $1+\frac{1}{\sqrt{5}}$.

(*a*) If the series is arithmetic, show that the common difference is $\frac{-4}{\sqrt{5}}$, and that the sum of the first six terms is $6-6\sqrt{5}$. **4**

(*b*) If the series is geometric, show that it has a sum to infinity, and that this sum is $\frac{1}{2}(5+3\sqrt{5})$. **4**

(a) $d=1+\frac{1}{\sqrt{5}}-(1+\sqrt{5})=-\sqrt{5}+\frac{1}{\sqrt{5}}=\frac{1}{\sqrt{5}}-\frac{5}{\sqrt{5}}=\frac{-4}{\sqrt{5}}$

$S_6=\frac{n}{2}(2a+(n-1)d)=\frac{6}{2}\left(2(1+\sqrt{5})+5\left(\frac{-4}{\sqrt{5}}\right)\right)$

$=6+6\sqrt{5}-\frac{60}{\sqrt{5}}=6+6\sqrt{5}-\frac{60\sqrt{5}}{5}$

$=6+6\sqrt{5}-12\sqrt{5}=6-6\sqrt{5}$

(b) $r=\frac{1+\frac{1}{\sqrt{5}}}{1+\sqrt{5}}=\frac{\frac{\sqrt{5}+1}{\sqrt{5}}}{1+\sqrt{5}}=\frac{\sqrt{5}+1}{\sqrt{5}}\times\frac{1}{1+\sqrt{5}}=\frac{\sqrt{5}+1}{\sqrt{5}+5}$

$-1<\frac{\sqrt{5}+1}{\sqrt{5}+5}<1$

$\Rightarrow$ ~~has a limit~~ sum to infinity

$S_\infty=\frac{a}{1-r}=\frac{1+\sqrt{5}}{1-\left(\frac{\sqrt{5}+1}{\sqrt{5}+5}\right)}=\frac{1+\sqrt{5}}{\frac{\sqrt{5}+5-\sqrt{5}-1}{\sqrt{5}+5}}=\frac{1+\sqrt{5}}{1}\times\frac{\sqrt{5}+5}{4}$

$=\frac{\sqrt{5}+5+5+5\sqrt{5}}{4}=\frac{6\sqrt{5}+10}{4}=\frac{3\sqrt{5}+5}{2}$

Starter 16.1.2017

Find the term independent of u in the expansion of $\left(\frac{2}{u^3} - 3u\right)^8$. 4

$\binom{8}{r}(2u^{-3})^{8-r}(-3u)^r$ $\quad \underline{u^0}$

$(2u^{-3})^{8-r}(-3u)^r$

$2^{8-r}u^{-24+3r}(-3^r u^r)$

$u^{-24+3r} \times u^r = u^0$

$-24+4r=0$

$4r=24$

$\underline{\underline{r=6}}$

$\binom{8}{6}(2u^{-3})^2(-3u)^6$

$(28)(4u^{-6})(729u^6)$

$= \underline{81648u^0}$

Recap. on Sigma Notation

Reminder: $\sum_{k=a}^{b} f(k)$ is shorthand for the sum

f(a) + f(a + 1) + f(a + 2) + ...+ f(b)

where a, b ∈ Z and a ≤ b

Evaluate the following:

1. $\sum_{k=1}^{3}(2k+1) = (2(1)+1) + (2(2)+1) + (2(3)+1) = 3+5+7 = \underline{\underline{15}}$

2. $\sum_{k=0}^{4}(-2)^k = -2^0 + (-2) + (-2)^2 + (-2)^3 + (-2)^4 = 1 - 2 + 4 - 8 + 16 = \underline{\underline{11}}$

3. $\sum_{k=2}^{5}(3k^2-7) = (3(2)^2-7) + (3(3)^2-7) + (3(4)^2-7) + (3(5)^2-7)$
$= 5 + 20 + 41 + 68 = \underline{\underline{134}}$

4. $\sum_{k=-1}^{2}(2k-5)(k-3) = (2(-1)-5)(-1-3) + (2(0)-5)(0-3) + (2(1)-5)(1-3)$
$+ (2(2)-5)(2-3) = (-7)(-4) + (-5)(-3) + (-3)(-2)$
$+ (-1)(-1) = 28 + 15 + 6 + 1$
$= \underline{\underline{50}}$

Using partial fractions to find the sum of a series

Partial fractions can be used to find the sum of a series.

Example:

(a) Express $\frac{1}{4k^2-1}$ in partial fractions

$= \frac{1}{(2k+1)(2k-1)} = \frac{A}{(2k+1)} + \frac{B}{(2k-1)}$

$A(2k-1) + B(2k+1)$

$2Ak - A + 2Bk + B$

$-A+B=1$ $\quad 2A+2B=0$

$-A-A=1$ $\quad A+B=0$

$-2A=1$ $\quad B=-A$

$\underline{A=-\frac{1}{2}}$ $\quad \underline{\underline{B=-(-\frac{1}{2})=\frac{1}{2}}}$

$\underline{\frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}}$

Using partial fractions to find the sum of a series

(b) Deduce that $\sum_{k=1}^{n}\frac{1}{4k^2-1} = \frac{1}{2} - \frac{1}{2(2n+1)}$

L.H.S

$\sum_{k=1}^{n} \frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}$

$= \underbrace{\frac{1}{2(2(1)-1)} - \frac{1}{2(2(1)+1)}}_{k=1} + \underbrace{\frac{1}{2(2(2)-1)} - \frac{1}{2(2(2)+1)}}_{k=2} + \underbrace{\frac{1}{2(2(3)-1)} - \frac{1}{2(2(3)+1)}}_{k=3}$

$+ \underbrace{\frac{1}{2(2(4)-1)} - \frac{1}{2(2(4)+1)}}_{k=4} \; \ldots\ldots \; + \underbrace{\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}}_{k=n}$

$= \underbrace{\frac{1}{2} - \cancel{\frac{1}{6}}} + \underbrace{\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}} + \underbrace{\cancel{\frac{1}{10}} - \cancel{\frac{1}{14}}} + \underbrace{\cancel{\frac{1}{14}} - \cancel{\frac{1}{18}}} \ldots\ldots$

$+ \underbrace{\cancel{\frac{1}{2(2n-1)}} - \frac{1}{2(2n+1)}}$

$= \underline{\underline{\frac{1}{2} - \frac{1}{2(2n+1)}}} = \text{R.H.S}$

Using partial fractions to find the sum of a series

Evaluate $\sum_{k=1}^{\infty}\frac{1}{4k^2-1}$

$\lim_{n \to \infty}\left(\frac{1}{2} - \frac{1}{2(2n+1)}\right)$

as $n \to \infty$

$-\frac{1}{2(2n+1)} \to 0$

Hence $\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \underline{\underline{\frac{1}{2}}}$

Use of Standard Formulae to find the sum of a series

First we will look at some properties of $\sum$

1. $\sum_{k=1}^{n}\{f(k)+g(k)\} = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$

Proof:

L.H.S

$\{f(1)+g(1)\} + \{f(2)+g(2)\} + \{f(3)+g(3)\} + \ldots\ldots + \{f(n)+g(n)\}$

$= f(1)+f(2)+f(3)+\ldots\ldots+f(n) + g(1)+g(2)+g(3)+\ldots\ldots+g(n)$

$= \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k) = \text{R.H.S}$

Starter 18.1.2017

$A=\begin{pmatrix} p & 1 \\ -2 & q \end{pmatrix}$, $B=\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$ and $AB=\begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$.

(a) Find the values of p and q. 2

(b) Hence find the matrix BA. 1

(c) A matrix C is such that $CAB = BA$. Find the matrix C. 2

(a) $\begin{pmatrix} p & 1 \\ -2 & q \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$

$\begin{pmatrix} 2p & p-1 \\ -4 & -2-q \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix}$

$2p=2$, $\underline{p=1}$

$-2-q=2$, $-q=4$, $\underline{q=-4}$

(b) $BA = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix}$

(c) $CAB = BA$

$\times (AB)^{-1}$ $\quad \times (AB)^{-1}$

$C = BA \times (AB)^{-1}$

$(AB)^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$

$C = \frac{1}{4}\begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$

$C = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$

Use of Standard Formulae to find the sum of a series

2. If a is a constant, then $\sum_{k=1}^{n} af(k) = a\sum_{k=1}^{n} f(k)$

Proof:

L.H.S

$af(1) + af(2) + af(3) + af(4) + \ldots + af(n)$

$= a\{f(1)+f(2)+f(3)+\ldots+f(n)\}$

$= a\sum_{k=1}^{n} f(k) = $ R.H.S

Use of Standard Formulae to find the sum of a series

If a and b are constants:

$$\sum_{k=1}^{n}\{af(k)+bg(k)\} = \sum_{k=1}^{n} af(k) + \sum_{k=1}^{n} bg(k)$$ from property 1

$$= a\sum_{k=1}^{n} f(k) + b\sum_{k=1}^{n} g(k)$$ from property 2.

This extends to more than two functions and holds for integration because integration is the limit of summation.

Use of Standard Formulae to find the sum of a series

The following formulae may be used to sum series. These are on your exam papers.

1. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Proof: L.H.S

$1+2+3+\ldots+(n-1)+n$

$S_n = \frac{n}{2}[2a+(n-1)d]$ $\quad a=1 \quad d=1$

$= \frac{n}{2}[2(1)+(n-1)1]$

$= \frac{n}{2}[2+n-1] = \frac{n}{2}[n+1] = \frac{n(n+1)}{2} = $ R.H.S

Use of Standard Formulae to find the sum of a series

2. $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$ Note also: $\sum_{r=1}^{n} r^3 = (\sum_{r=1}^{n} r)^2$

These can be proved using Induction.

Use of Standard Formulae to find the sum of a series

Examples:

1. Evaluate $\sum_{r=1}^{10} r = \frac{n(n+1)}{2} = \frac{10(10+1)}{2} = \underline{\underline{55}}$

2. Evaluate $\sum_{r=1}^{8} 2r = 2\sum_{r=1}^{8} r = 2\left(\frac{n(n+1)}{2}\right) = 2\left(\frac{8(8+1)}{2}\right) = \underline{\underline{72}}$

Use of Standard Formulae to find the sum of a series

Examples:

3. Evaluate $\sum_{r=1}^{6}(r+3) = \sum_{r=1}^{6} r + 6(3)$

1+3+ 2+3+3+3 +4+3+ ....

$= \frac{6(6+1)}{2} + 18 = 21 + 18 = \underline{\underline{39}}$

4. Evaluate $\sum_{r=1}^{5}(2r-3) = 2\sum_{r=1}^{5} r + 5(-3)$

$= 2\left[\frac{5(5+1)}{2}\right] - 15 = \underline{\underline{15}}$

$\sum_{k=1}^{10}(2k^3-3k) = 2\sum_{k=1}^{10} k^3 - 3\sum_{k=1}^{10} k$

$= 2\left[\frac{n^2(n+1)^2}{2}\right] - 3\left[\frac{n(n+1)}{2}\right]$

$= 2\left[\frac{10^2(10+1)^2}{4}\right] - 3\left[\frac{10(10+1)}{2}\right] =$ $6050 - 165 = \underline{\underline{5885}}$

Use of Standard Formulae to find the sum of a series

1. Find an expression for $\sum_{k=1}^{n} k(k+1)$, giving your answer in fully factorised form.
   Hence sum the series $(1\times 2)+(2\times 3)+(3\times 4)+(4\times 5)+\ldots+(28\times 29)$.

2. Find an expression for $\sum_{k=1}^{n} k(k+3)$, giving your answer in fully factorised form.
   Hence sum the series $(1\times 4)+(2\times 5)+(3\times 6)+(4\times 7)+\ldots+(35\times 38)$.